Google Research

Stochastic facility location problems with outsourcing costs

Eduardo Moreno (Google Research, France)
Ivana Ljubić (ESSEC Business School of Paris, France)
Javiera Barrera (Universidad Adolfo Ibáñez, Chile)

JFCO 2025, Jul 8-11, 2005.

Problem definition: a two-stage stochastic problem

- **First stage**: an "assignment" problem.
  - Facility location: Customers assigned to open facilities
  - Generalized Assignment: Tasks assigned to agents
  - Vehicle Routing: Customers assigned to vehicles
- **Second stage**: an stochastic demand to be served with outsourcing/penalty.
  - Customer/task demands are served by the assigned facility/agent/vehicle minimizing cost.
  - If the total demand is higher than the capacity, the unserved demands is outsourced / penalized at a higher cost.
- **Objective**: Minimize the assignment cost + expected value of serving demand.
- **This talk:** Solve the problem for general probability distributions (not scenarios)

Capacitated Facility Location

$$\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij}$$

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le d_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0,d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i

Google Research

Capacitated Facility Location

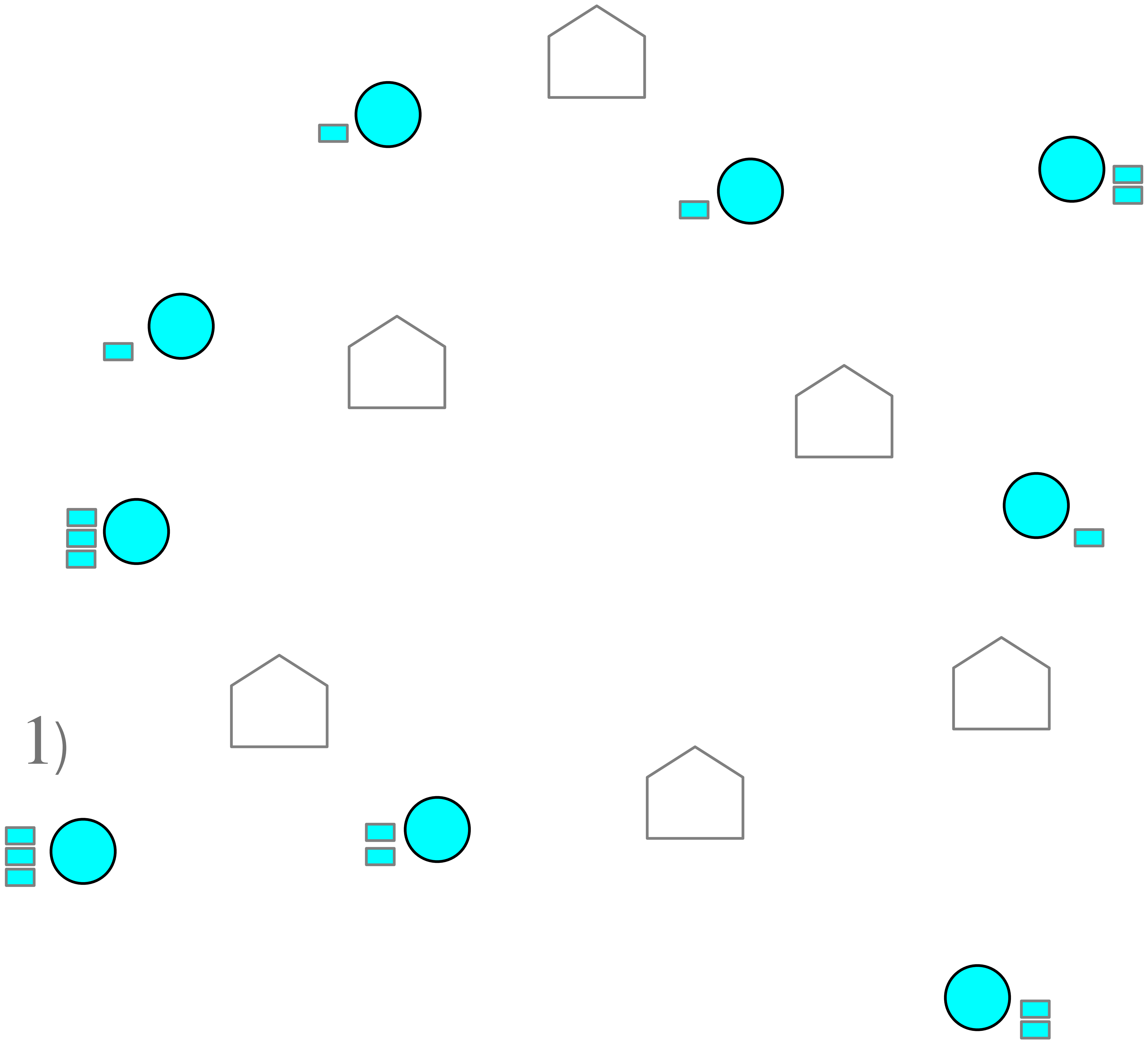
$$\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij}$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq d_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0, d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i



Google Research

Capacitated Facility Location

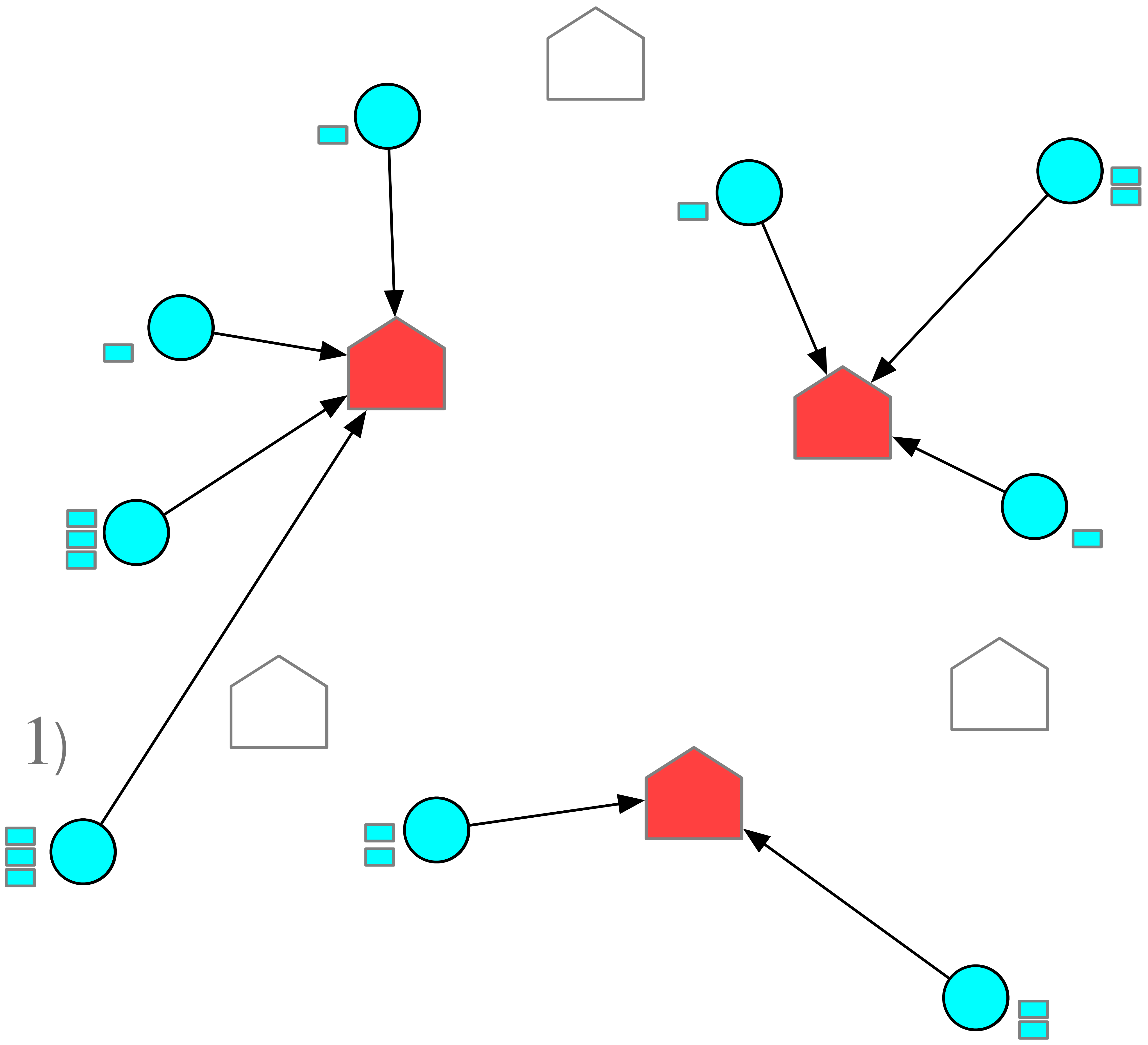
$$\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij}$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq d_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0,d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i



Google Research

Capacitated Facility Location

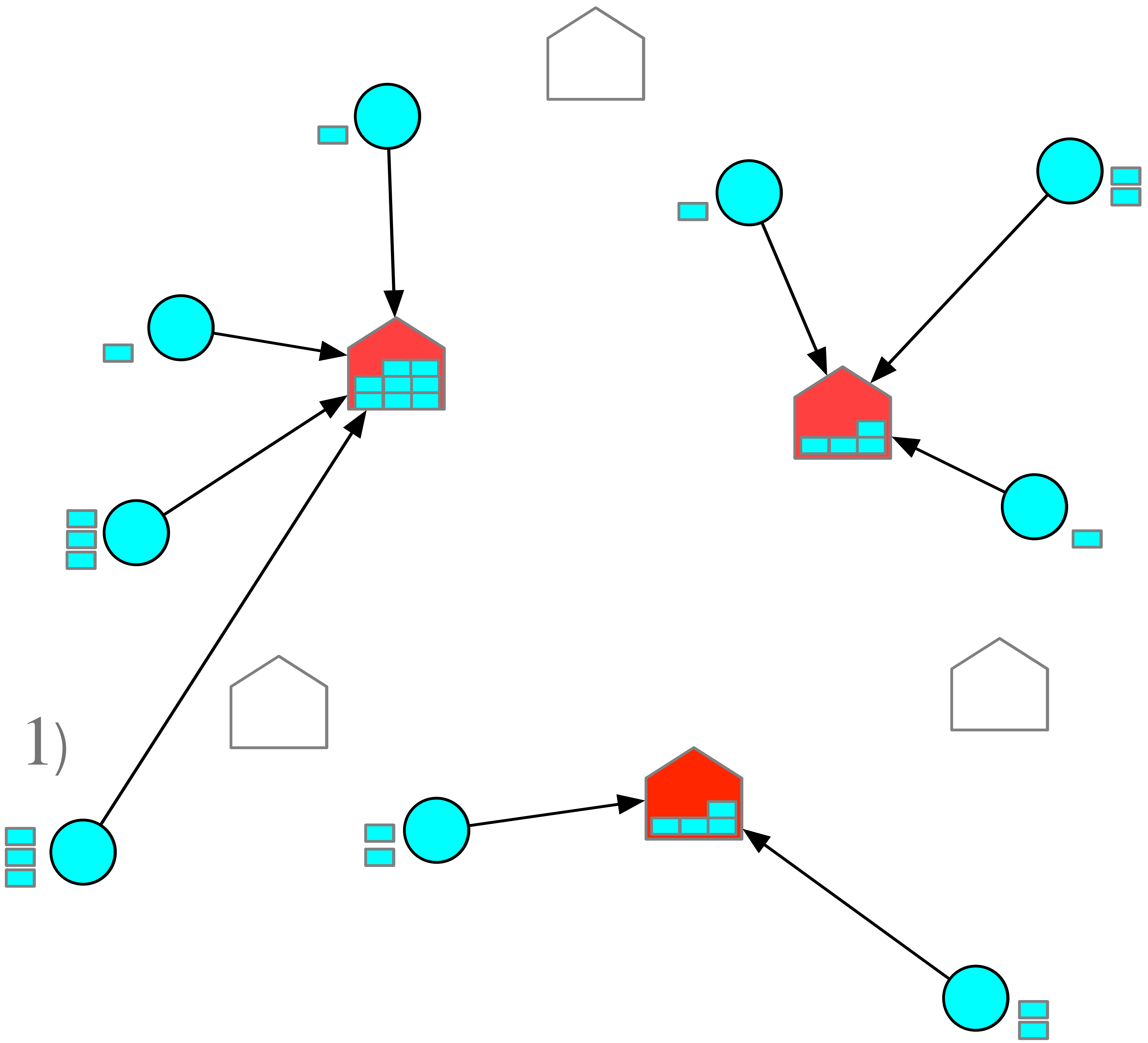
$$\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij}$$

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le d_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0,d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i



Google Research

Capacitated Facility Location

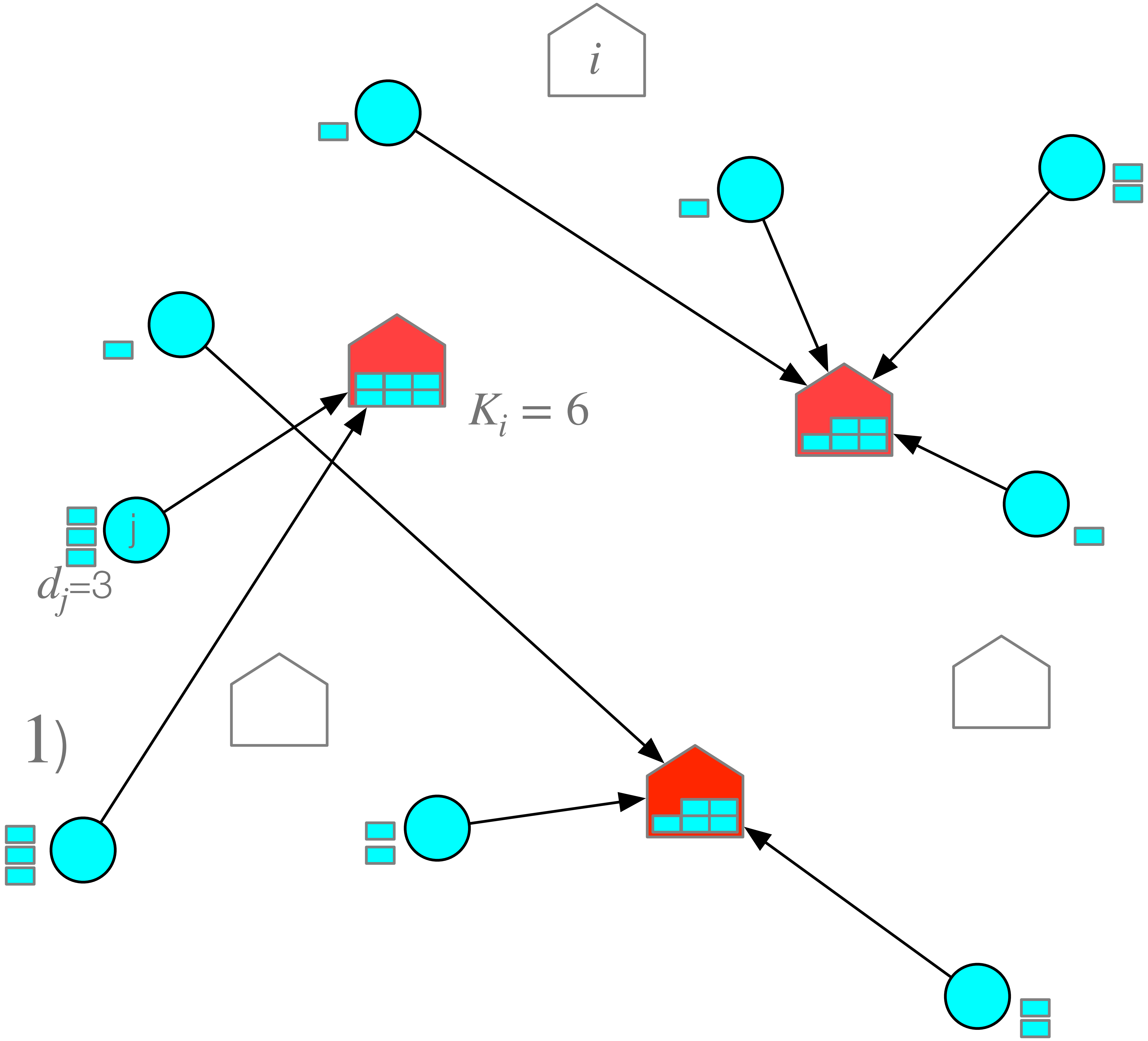
$$\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij}$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq d_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0,d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i



Google Research

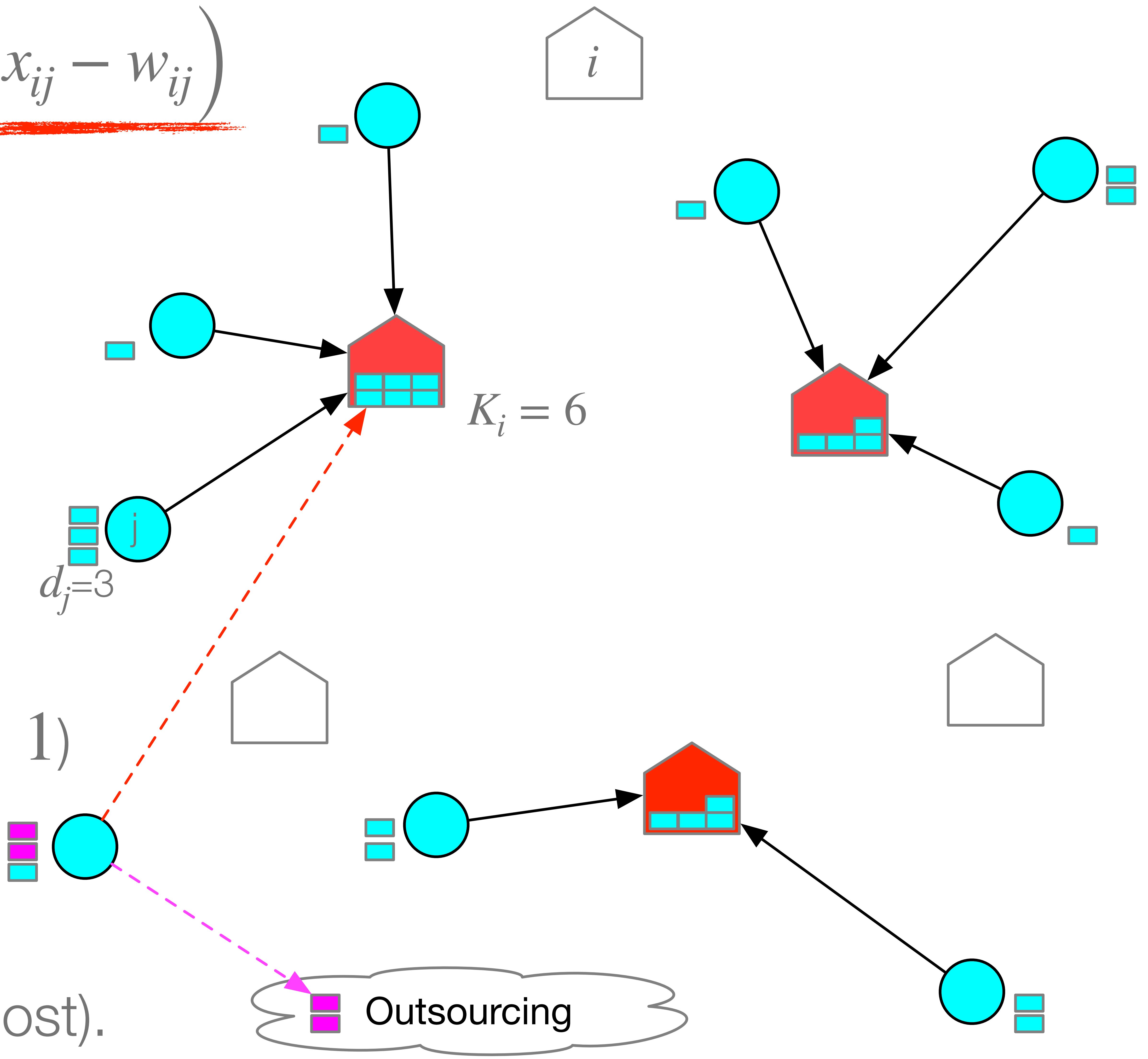
Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij} + g_{ij} \left(d_j \cdot x_{ij} - w_{ij} \right)$$

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le d_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$



Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0, d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i
allowing to outsource some demand (at a higher cost).

Capacitated Facility Location with Outsourcing

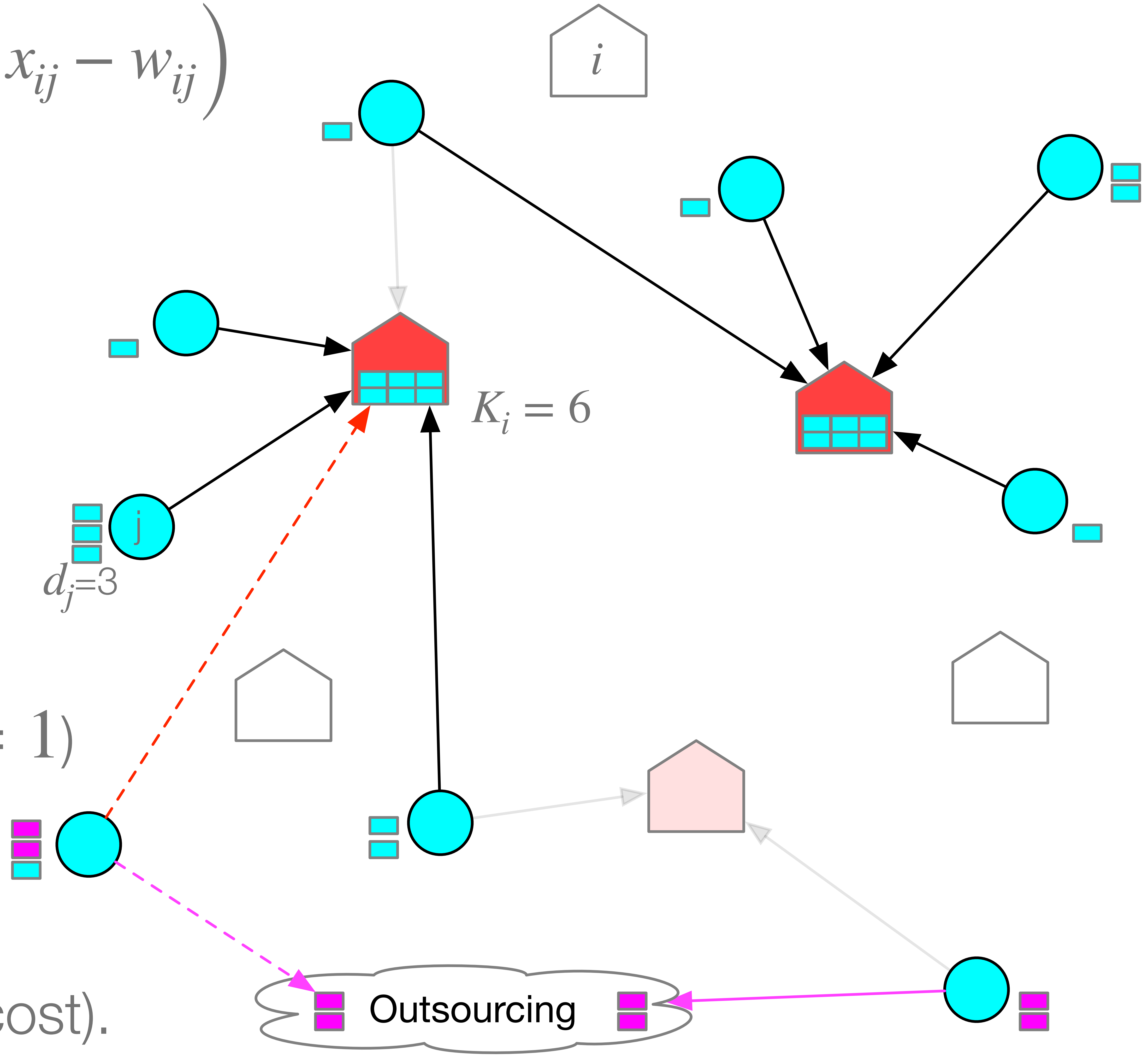
$$\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij} + g_{ij}\left(d_j \cdot x_{ij} - w_{ij}\right)$$

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le d_j \cdot x_{ij}$$

$$(x, y) \in \mathscr{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0,d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i
allowing to outsource some demand (at a higher cost).



Stochastic Capacitated Facility Location with Outsourcing

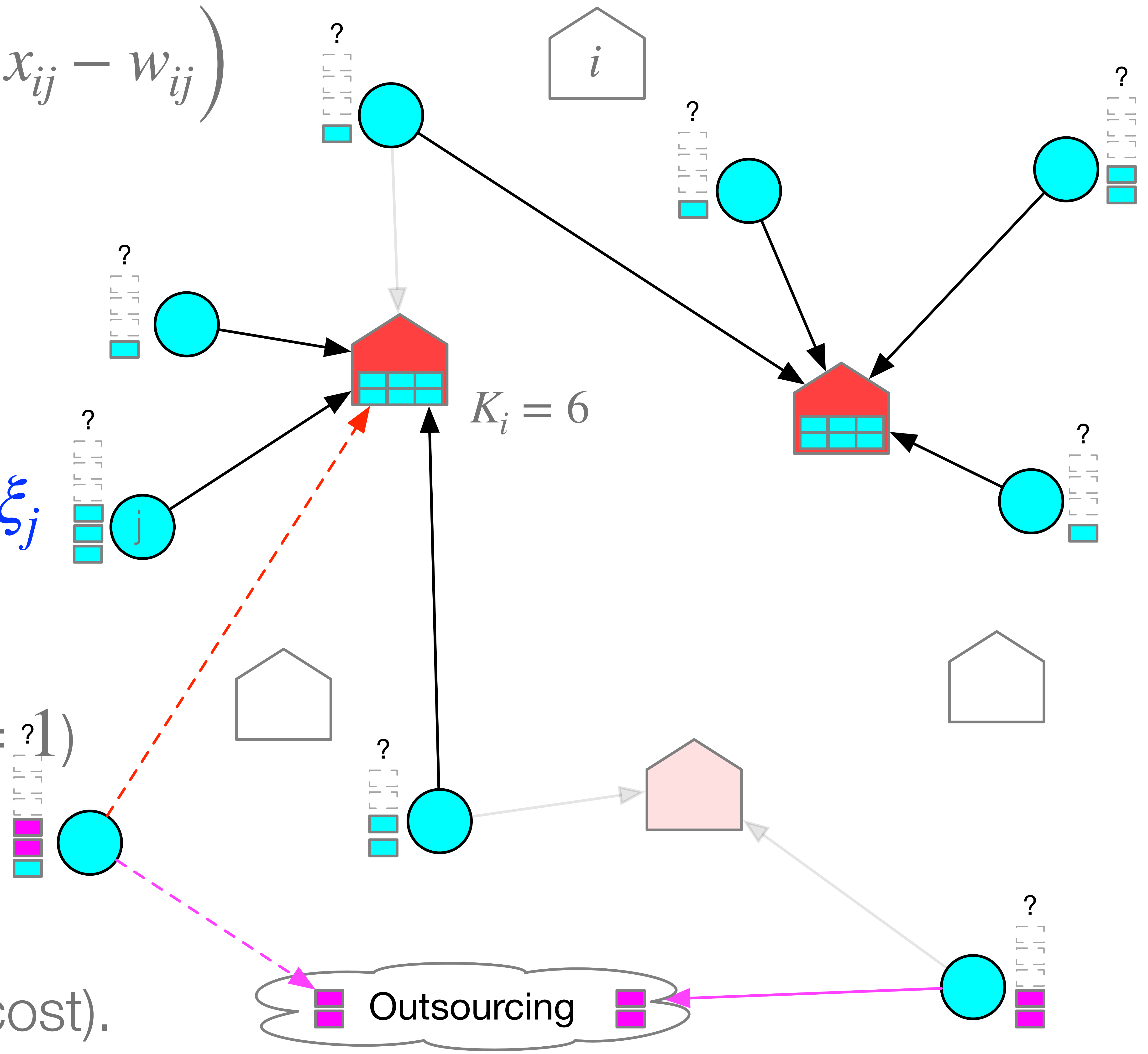
$$\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij} + g_{ij} \left(\xi_j \cdot x_{ij} - w_{ij} \right)$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathscr{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with random demand ξ_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0, d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i
allowing to outsource some demand (at a higher cost).



Google Research

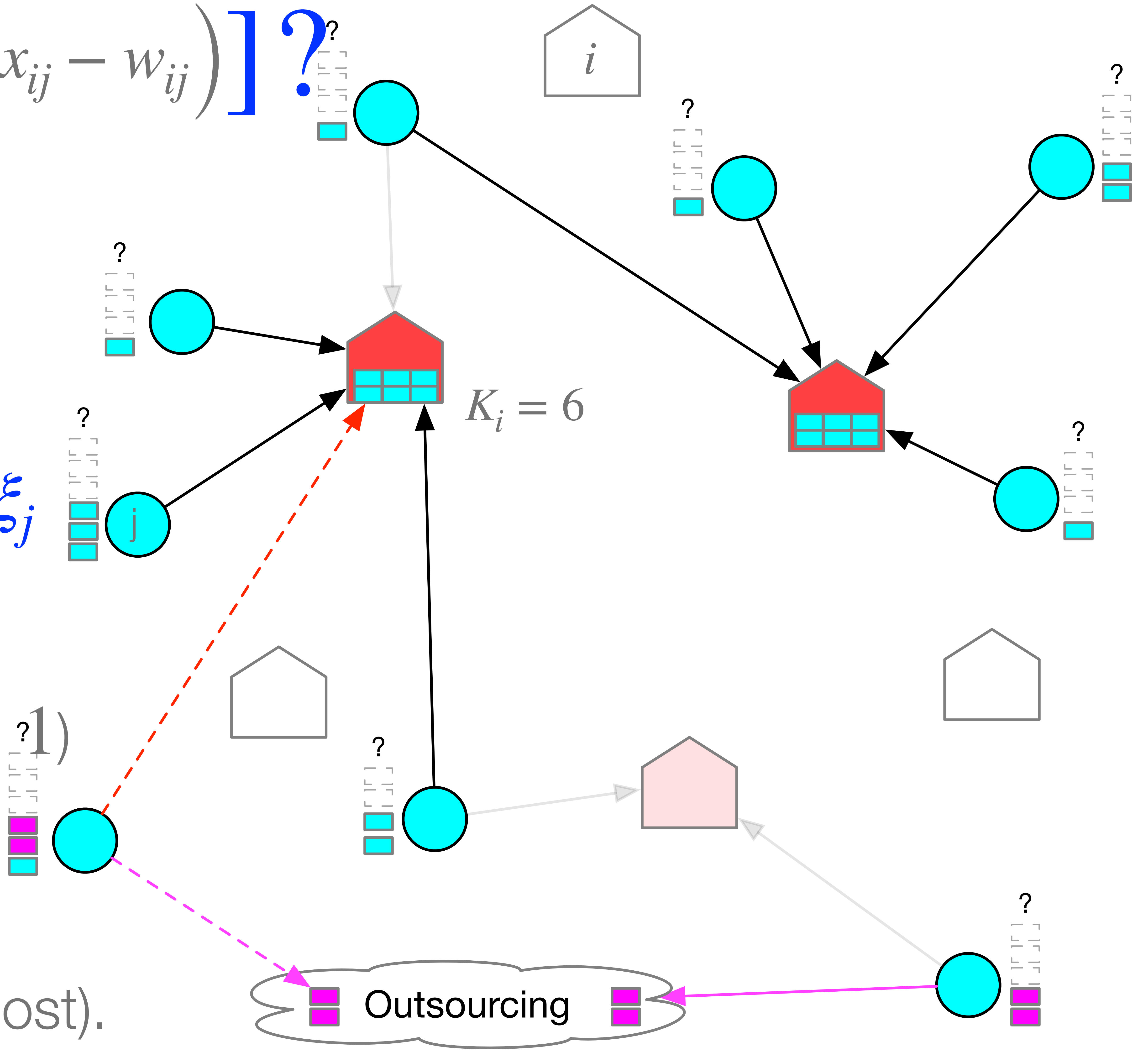
Stochastic Capacitated Facility Location with Outsourcing

$$\mathbb{E}\left[\min \sum_i f_i y_i + \sum_i \sum_j c_{ij} w_{ij} + g_{ij}\left(\xi_j \cdot x_{ij} - w_{ij}\right)\right] ?$$

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$



Given a set of customers J with random demand ξ_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients $(w_{ij} \in [0, d_j])$
minimizing the installment and assignment cost.
while satisfying the capacity of the facility K_i
allowing to outsource some demand (at a higher cost).

Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mathbb{E}\left[c_{ij} w_{ij} + g_{ij}\left(\xi_j \cdot x_{ij} - w_{ij}\right)\right]$$

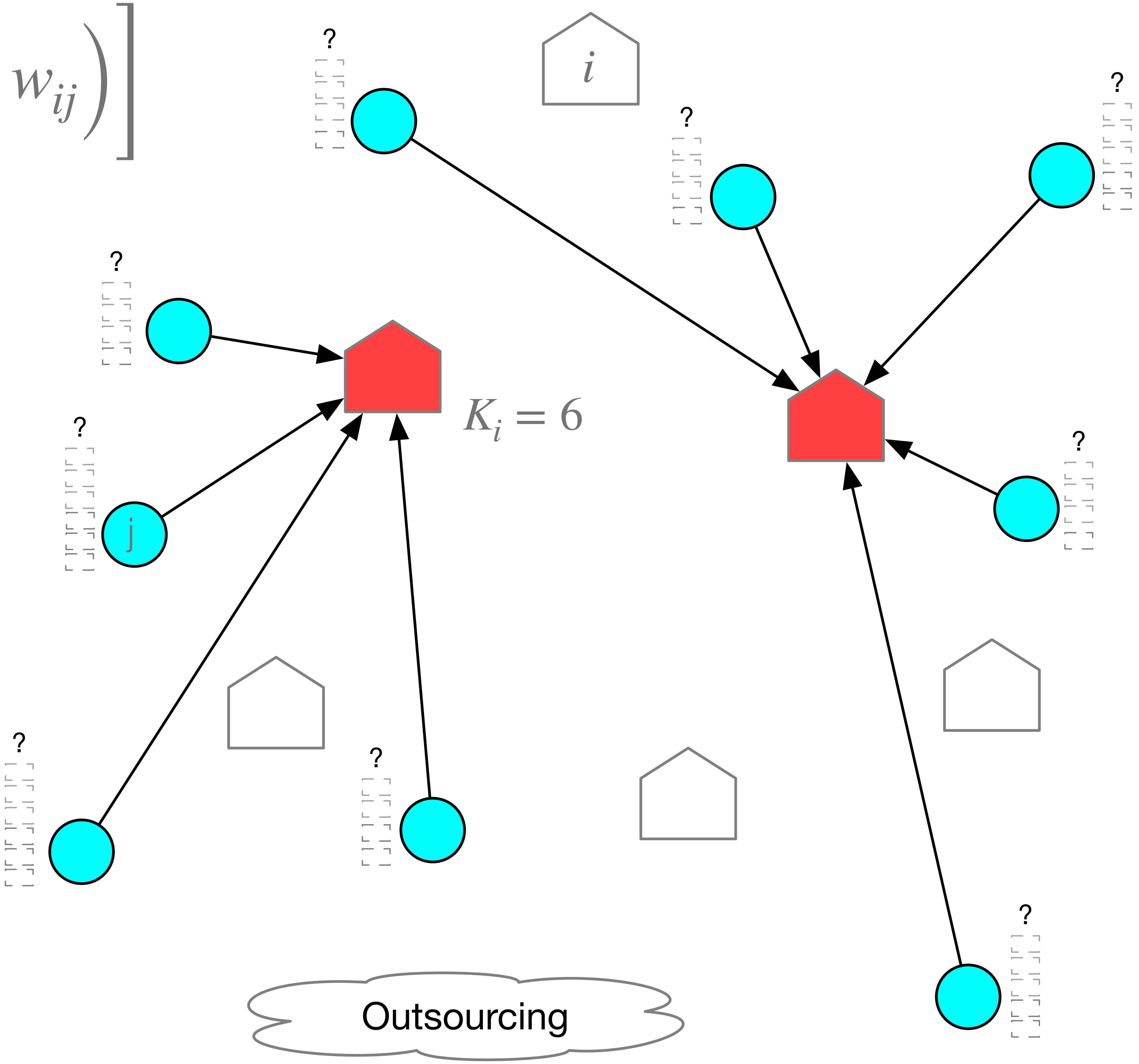
$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.



Google Research

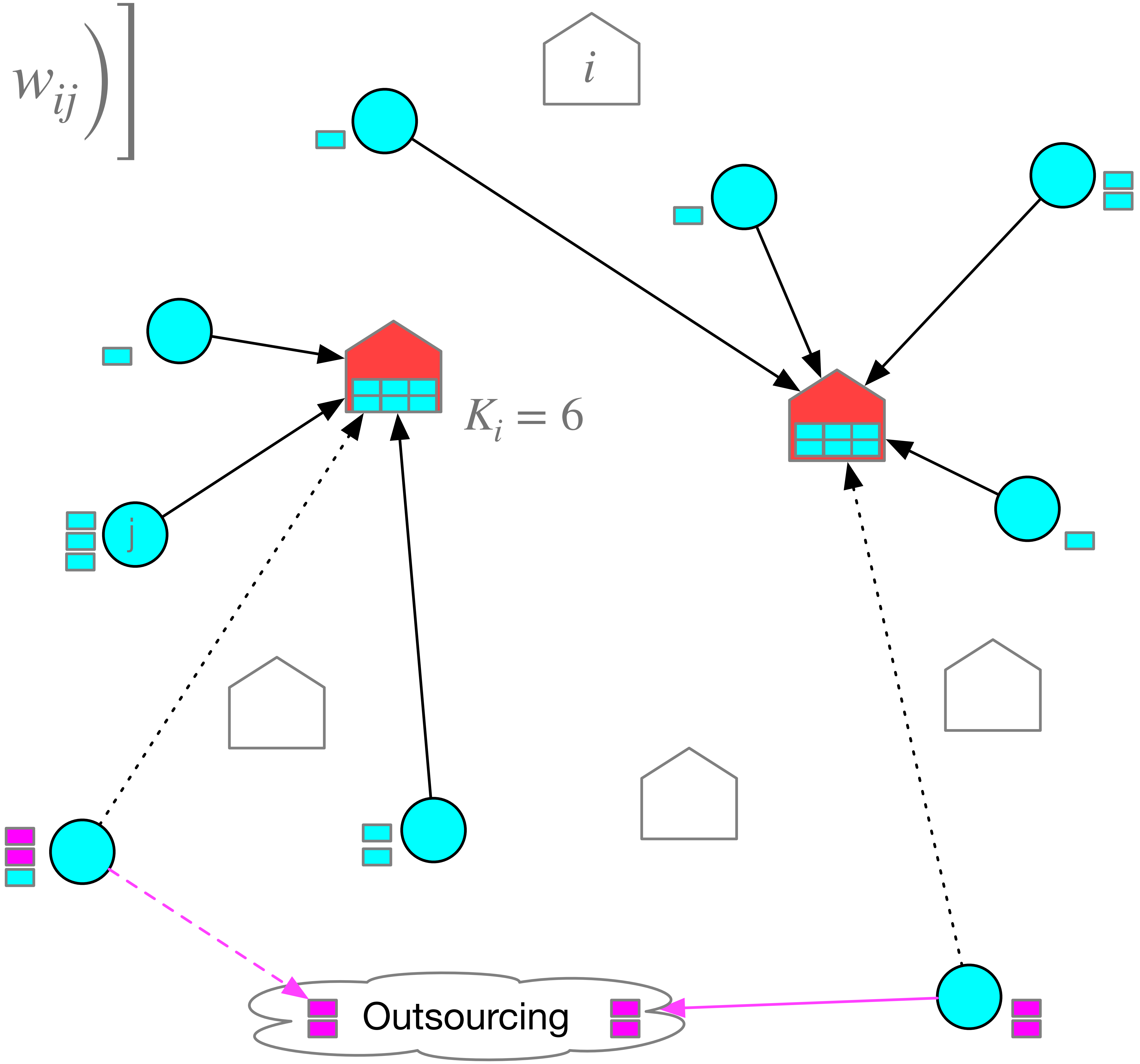
Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mathbb{E}\left[c_{ij} w_{ij} + g_{ij}\left(\xi_j \cdot x_{ij} - w_{ij}\right)\right]$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathscr{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$



We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.
- **2nd stage decision (wait-and-see):** to route and/or outsource the random demand.

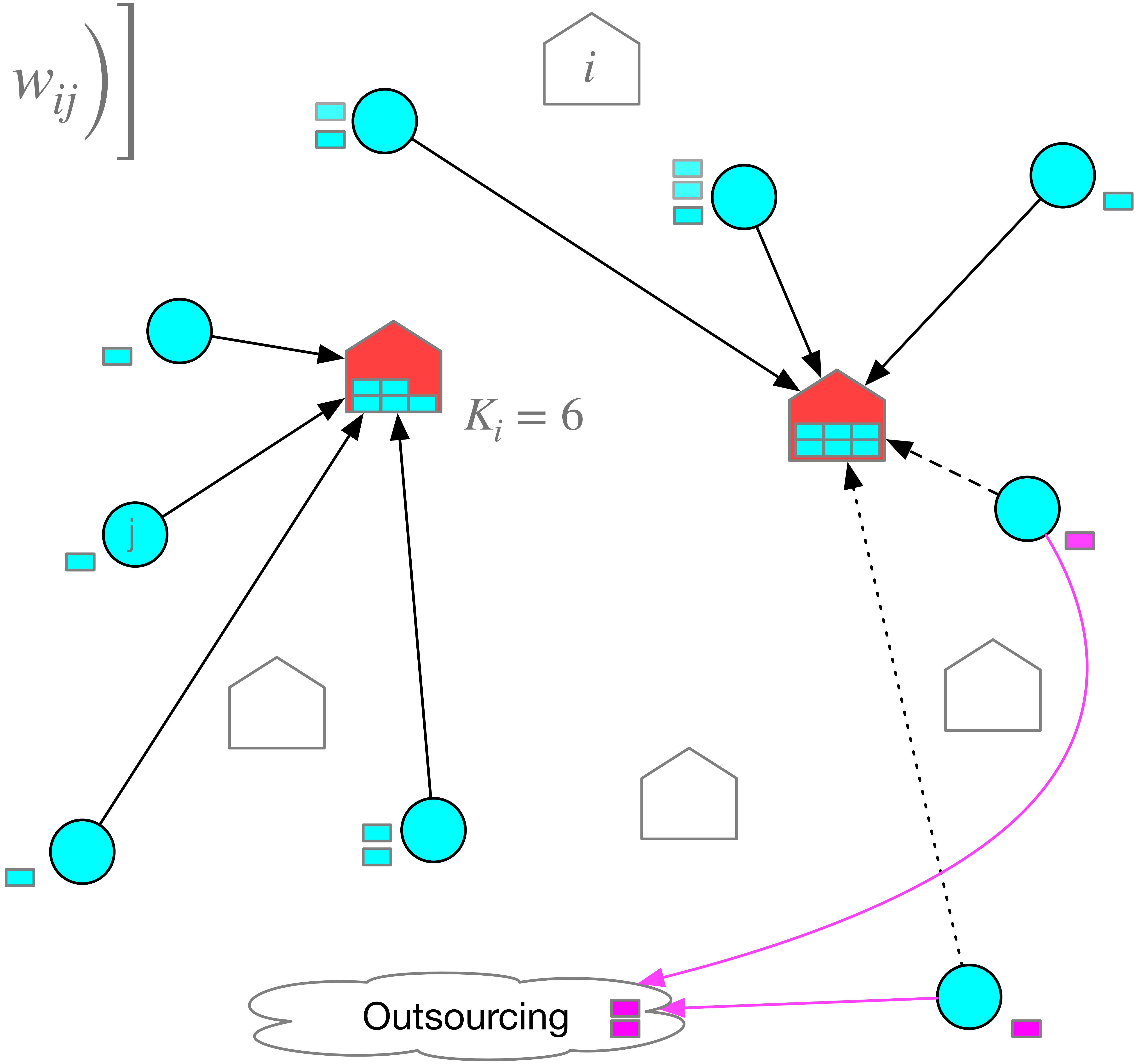
Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mathbb{E}\left[c_{ij} w_{ij} + g_{ij}\left(\xi_j \cdot x_{ij} - w_{ij}\right)\right]$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$



We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.
- **2nd stage decision (wait-and-see):** to route and/or outsource the random demand.

Google Research

Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mu_j g_{ij} x_{ij} - \sum_i \mathbb{E}\left[\sum_j (g_{ij} - c_{ij}) w_{ij}\right]$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.
- **2nd stage decision (wait-and-see):** to route and/or outsource the random demand.

Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mu_j g_{ij} x_{ij} - \sum_i \mathbb{E}[Q^i(x, y, \xi)]$$

$$x_{ij} \le y_i$$

$$(x, y) \in \mathcal{X}$$

$$x, y \in \{0,1\}$$

First-stage problem

$$Q^i(x, y, \xi) = \max_{w \ge 0} \sum_j (g_{ij} - c_{ij}) w_{ij}$$

$$w_{ij} \le \xi_j \cdot x_{ij}$$

$$\sum_j w_{ij} \le K_i \cdot y_i$$

Second-stage problem (independent for each facility)

We assume a two-stage stochastic problem:

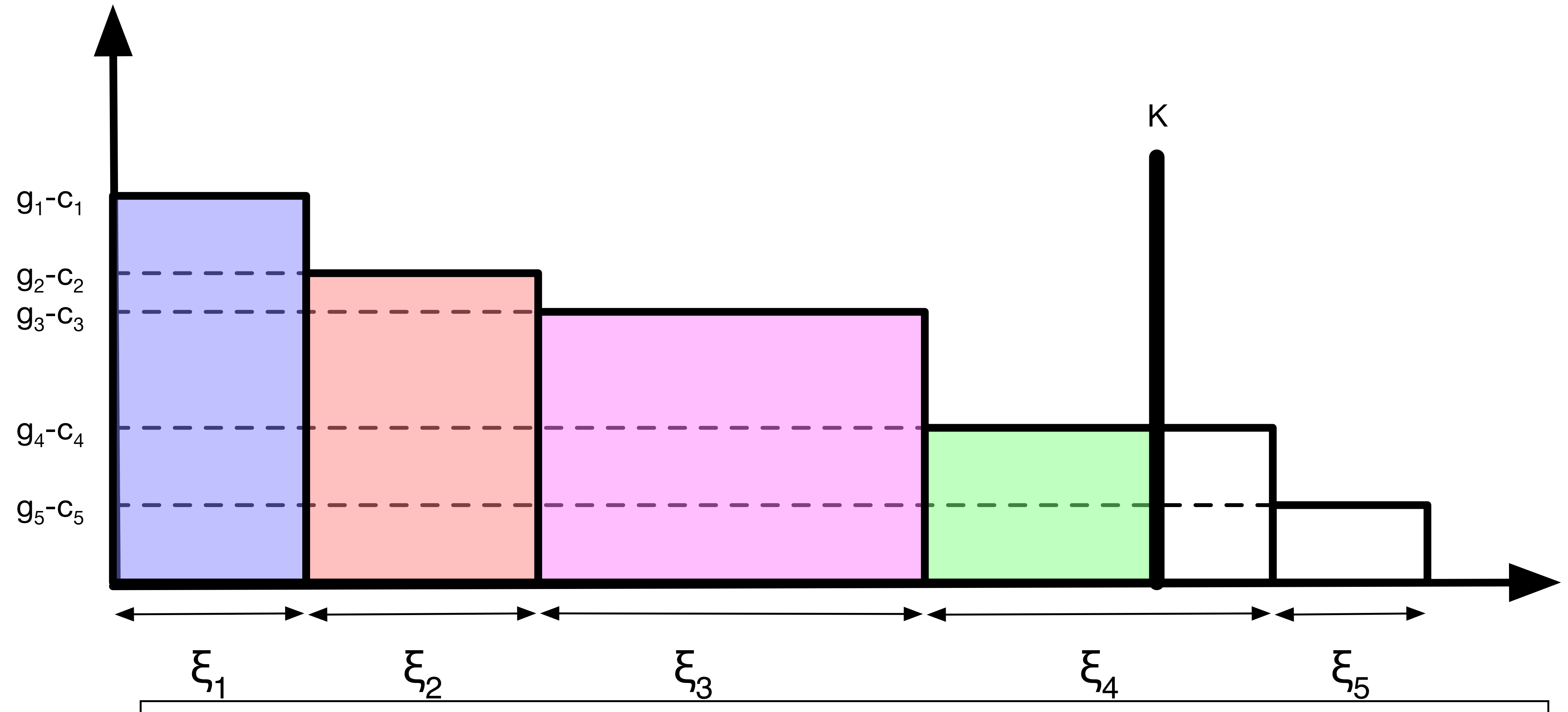
- **1st stage decision (here-and-now):** to open facilities and assign customers to them.
- **2nd stage decision (wait-and-see):** to route and/or outsource the random demand.

Second-stage problem : Knapsack problem

$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j (g_{ij} - c_{ij}) w_{ij}$$

$$w_{ij} \leq \xi_j \cdot x_{ij}$$

$$\sum_j w_{ij} \leq K_i \cdot y_i$$



Notation assumption: indices j are already in decreasing order of profit.

Optimal solution: to allocate the demand in decreasing order of profit $g_{ij} - c_{ij}$ until the capacity of the facility is fulfilled.

$$w_{ij} = \begin{cases} \xi_j x_{ij} & j < \tau^i \\ K_i y_i - \sum_{l < \tau^i} x_{il} \xi_l & j = \tau^i \\ 0 & j > \tau^i \end{cases}$$

Stochastic facility location problems with outsourcing costs

1. **Bender formulation for a <u>discrete set of scenarios</u> (for example, a sample average approximation of the demand distributions)**
2. Bender formulation for general distributions.
3. Computational experiments

Dual solution of the subproblem

Primal formulation:

$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j (g_{ij} - c_{ij}) w_{ij}$$
$$w_{ij} \leq \xi_j \cdot x_{ij} \qquad \forall j \in J$$
$$\sum_j w_{ij} \leq K_i \cdot y_i$$

The dual of the subproblem is given by

$$\hat{Q}^i(x, y, \xi) := \min_{\alpha, \gamma} \sum_{j \in J} \alpha_{ij} \xi_j x_{ij} + \gamma_i K_i y_i$$
$$\alpha_{ij} + \gamma_i \geq g_{ij} - c_{ij} \qquad \forall j \in J$$
$$\alpha_{ij}, \gamma_i \geq 0 \qquad \forall j \in J$$

Dual solution of the subproblem

The dual of the subproblem is given by

$$\hat{Q}^i(x,y,\xi) := \min_{\alpha,\gamma} \sum_{j\in J} \alpha_{ij}\xi_j x_{ij} + \gamma_i K_i y_i$$

$$\alpha_{ij} + \gamma_i \geq g_{ij} - c_{ij} \qquad \forall j \in J$$

$$\alpha_{ij}, \gamma_i \geq 0 \qquad \forall j \in J$$

and its solution is given by

$$\gamma_i = \begin{cases} \hat{v}_i^\xi & y_i^* > 0 \\ 0 & y_i^* = 0 \end{cases} \qquad \alpha_{ij} = (g_{ij} - c_{ij} - \hat{v}_i^\xi)^+ \quad \forall j \in J$$

and $\hat{v}_i^\xi$ is the cost of the critical customer τ^i where the capacity of the facility is fulfilled (or zero if not).

Hence: $$\hat{Q}^i(x,y,\xi) = \sum_{j\in J} \left(g_{ij} - c_{ij} - \hat{v}_i^\xi\right)^+ \cdot \xi_j \cdot x_{ij} + \hat{v}_i^\xi \cdot K_i \cdot y_i$$

Google Research

Dual solution of the subproblem

The dual of the subproblem is given by

$$\hat{Q}^i(x, y, \xi) := \min_{\alpha,\gamma} \sum_{j \in J} \alpha_{ij} \xi_j x_{ij} + \gamma_i K_i y_i$$

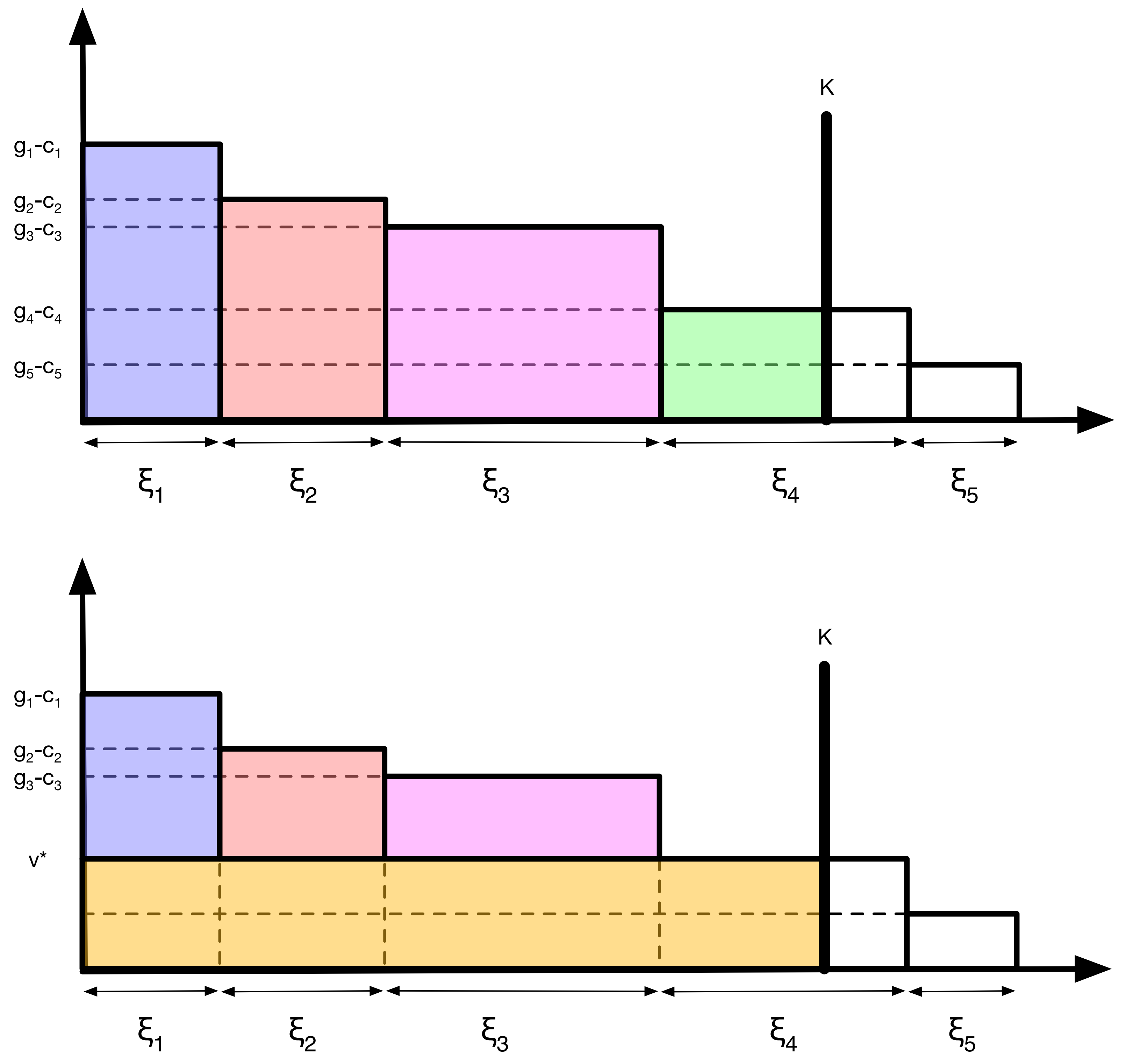
$$\alpha_{ij} + \gamma_i \geq g_{ij} - c_{ij} \qquad \forall j \in J$$

$$\alpha_{ij}, \gamma_i \geq 0 \qquad \forall j \in J$$

and its solution is given by

$$\gamma_i = \begin{cases} \hat{v}_i^{\xi} & y_i^* > 0 \\ 0 & y_i^* = 0 \end{cases} \qquad \alpha_{ij} = (g_{ij} - c_{ij} - \hat{v}_i^{\xi})^+ \quad \forall j \in J$$

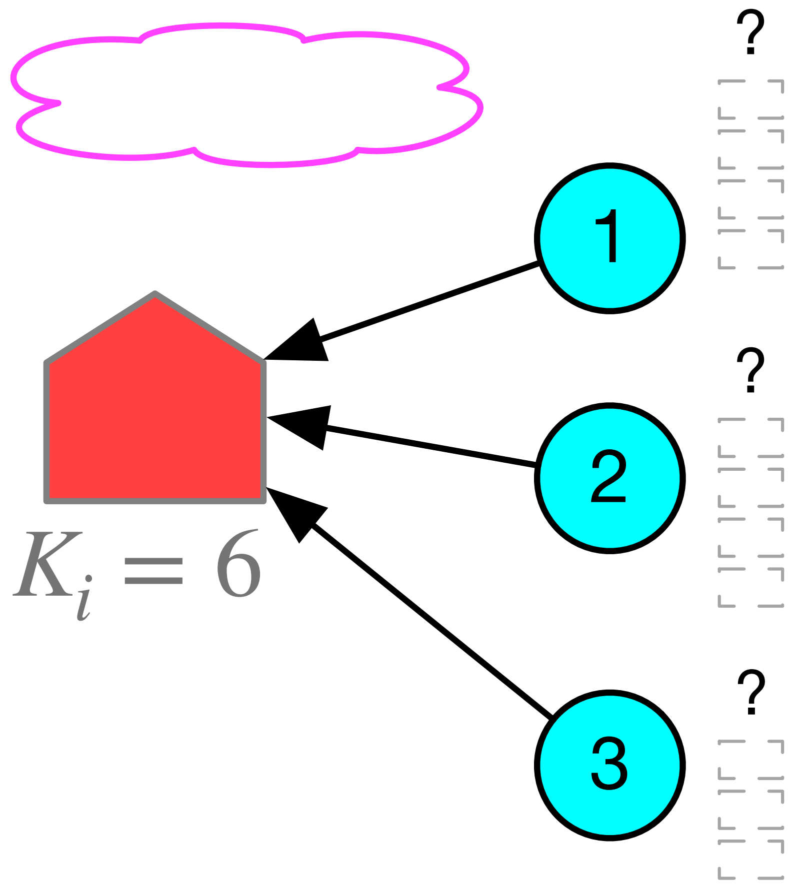
and $\hat{v}_i^{\xi}$ is the cost of the critical customer τ^i where the capacity of the facility is fulfilled (or zero if not).



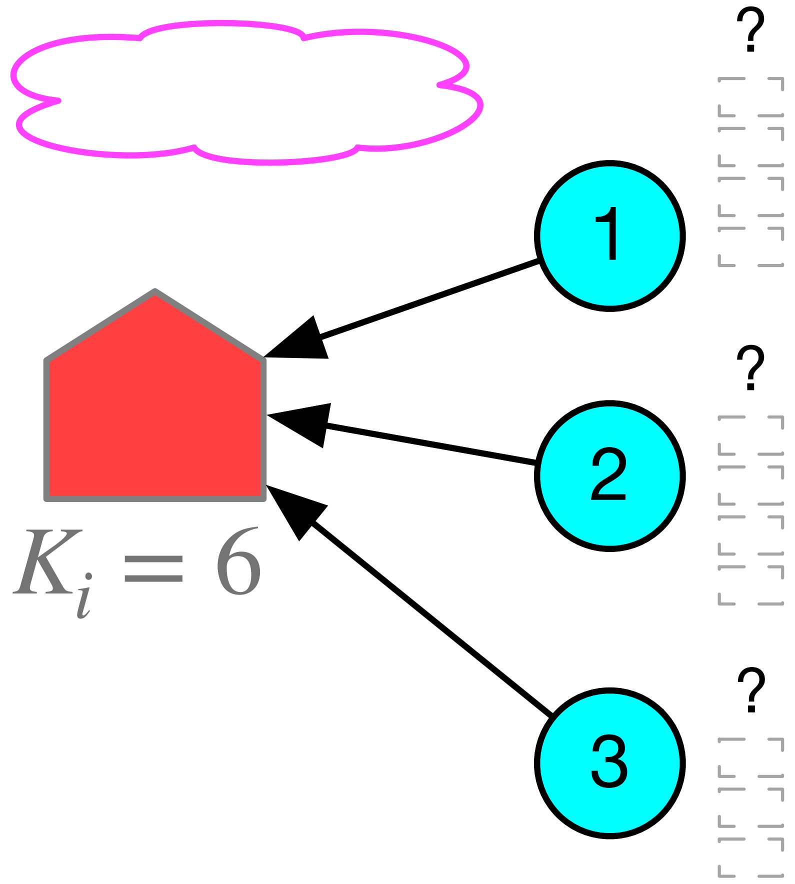
Hence: $$\hat{Q}^i(x, y, \xi) = \sum_{j \in J} \left(g_{ij} - c_{ij} - \hat{v}_i^{\xi}\right)^+ \cdot \xi_j \cdot x_{ij} + \hat{v}_i^{\xi} \cdot K_i \cdot y_i$$

Google Research

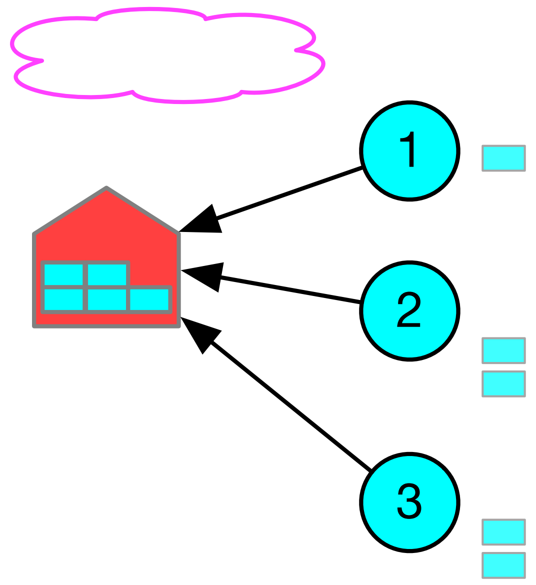
Dual solution of the subproblem



Dual solution of the subproblem



Scenario 1



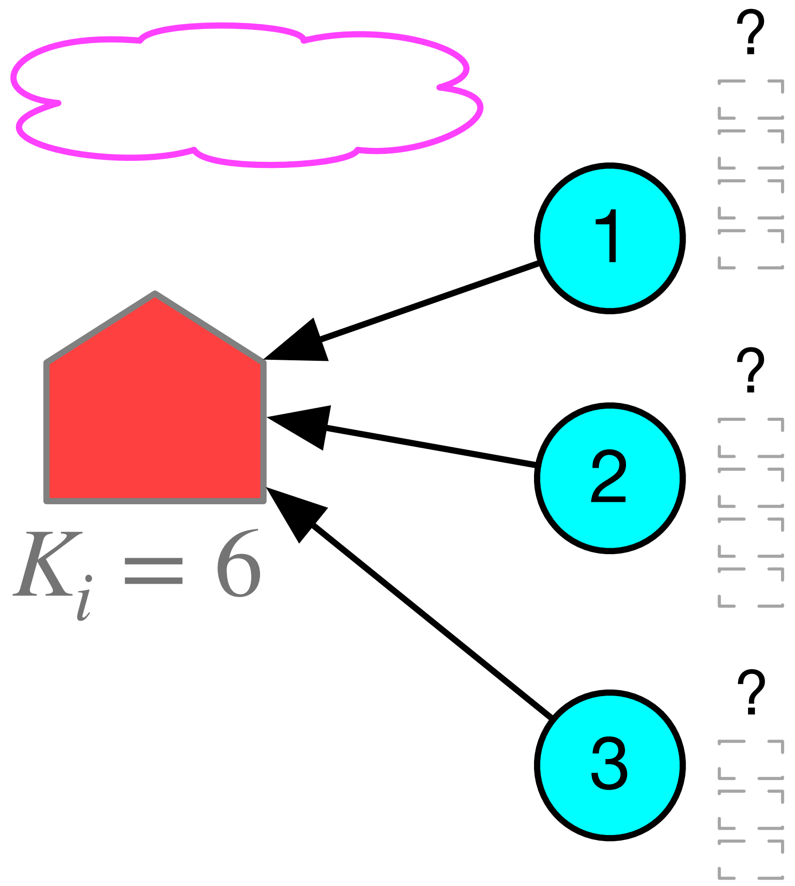
Primal solution

$$w_{i1} = 1$$
$$w_{i2} = 2$$
$$w_{i3} = 2$$

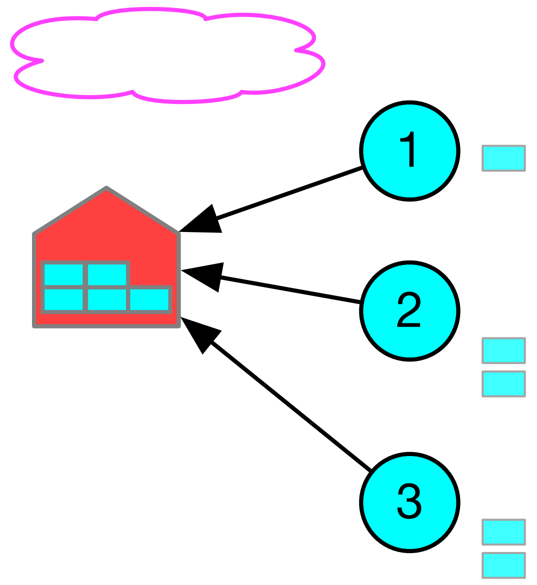
Dual solution

$$\tau^i = \varnothing \Rightarrow \hat{v}_i = 0$$
$$\alpha_{i1} = g_{i1} - c_{i1}$$
$$\alpha_{i2} = g_{i2} - c_{i2}$$
$$\alpha_{i3} = g_{i3} - c_{i3}$$

Dual solution of the subproblem



Scenario 1



Primal solution

$w_{i1} = 1$

$w_{i2} = 2$

$w_{i3} = 2$

Dual solution

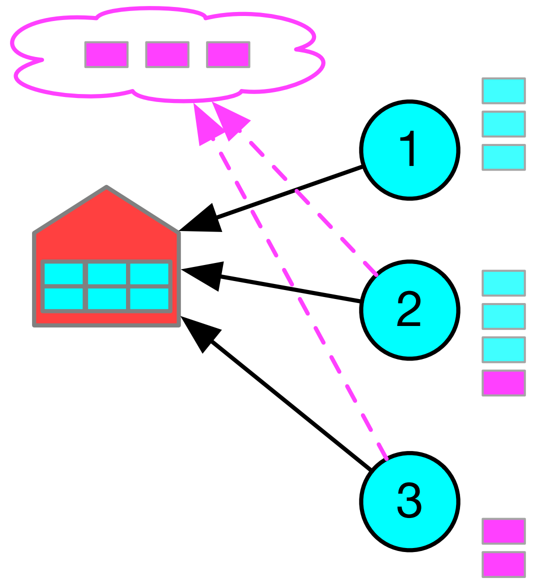
$\tau^i = \varnothing \Rightarrow \hat{v}_i = 0$

$\alpha_{i1} = g_{i1} - c_{i1}$

$\alpha_{i2} = g_{i2} - c_{i2}$

$\alpha_{i3} = g_{i3} - c_{i3}$

Scenario 2



Primal solution

$w_{i1} = 3$

$w_{i2} = 3$

$w_{i3} = 0$

Dual solution

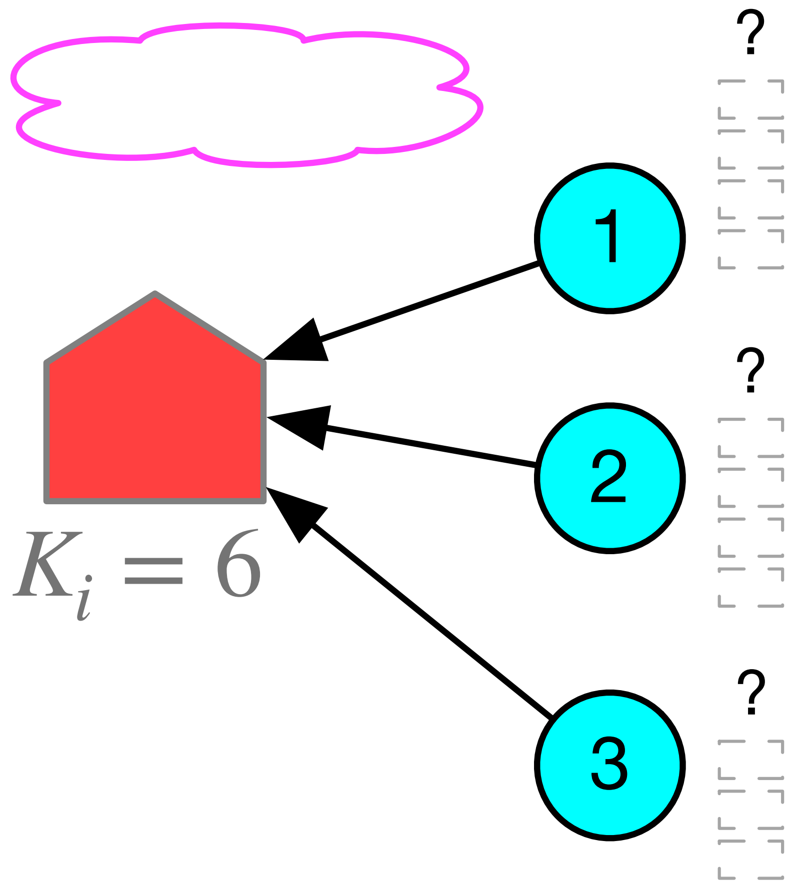
$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$

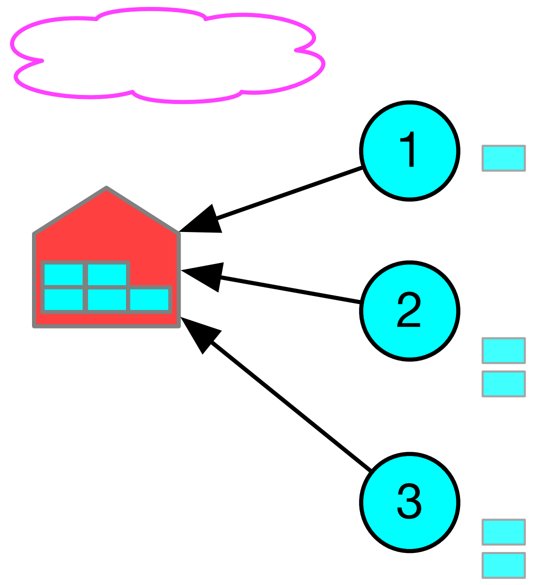
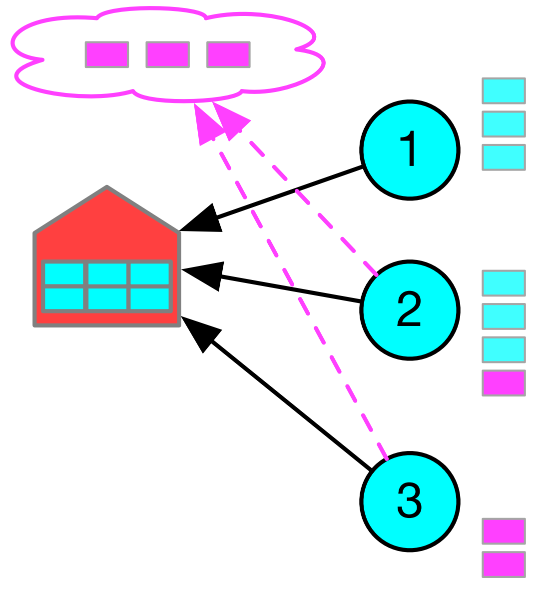
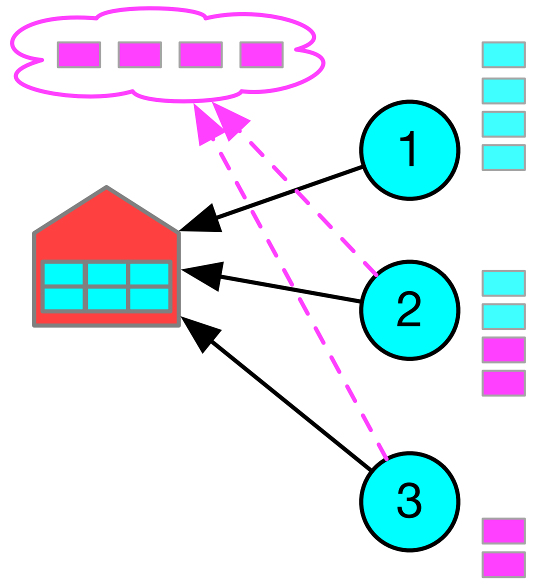
$\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$

$\alpha_{i2} = 0$

$\alpha_{i3} = 0$

Dual solution of the subproblem



Scenario		Primal solution	Dual solution
Scenario 1		$w_{i1} = 1$ $w_{i2} = 2$ $w_{i3} = 2$	$\tau^i = \varnothing \Rightarrow \hat{v}_i = 0$ $\alpha_{i1} = g_{i1} - c_{i1}$ $\alpha_{i2} = g_{i2} - c_{i2}$ $\alpha_{i3} = g_{i3} - c_{i3}$
Scenario 2		$w_{i1} = 3$ $w_{i2} = 3$ $w_{i3} = 0$	$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$ $\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$ $\alpha_{i2} = 0$ $\alpha_{i3} = 0$
Scenario 3		$w_{i1} = 4$ $w_{i2} = 2$ $w_{i3} = 0$	$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$ $\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$ $\alpha_{i2} = 0$ $\alpha_{i3} = 0$

Benders formulation for a discrete set of scenarios

Given a discrete set of scenarios $s \in S$ with probability p_s, we can reformulate

$$\min_{x,y} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} g_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{s \in S} p_s \theta_s^i$$

$$(x, y) \in \mathscr{X} \qquad x, y \in \{0,1\}$$

$$\theta_s^i \leq \hat{Q}^i(x, y, \xi^s) \qquad \forall i \in I, s \in S$$

and

$$\hat{Q}^i(x, y, \xi^s) \leq \sum_{j \in J} \left(g_{ij} - c_{ij} - \hat{v}_i^\xi \right)^+ \cdot \xi_j \cdot x_{ij}^* + \hat{v}_i^\xi \cdot K_i \cdot y_i^*$$

for any feasible dual solution

Benders formulation for a discrete set of scenarios

Given a discrete set of scenarios $s \in S$ with probability p_s, we can reformulate

$$\min_{x,y} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} g_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{s \in S} p_s \theta_s^i$$

$$(x, y) \in \mathcal{X} \qquad x, y \in \{0,1\}$$

$$\theta_s^i \leq \hat{Q}^i(x, y, \xi^s) \qquad \forall i \in I, s \in S$$

We can be solve it by iteratively adding Benders optimality cuts for the given incumbent solution (x^*, y^*)

$$\theta_s^i \leq \sum_{j \in J} \left(g_{ij} - c_{ij} - \hat{v}_i^{\xi} \right)^+ \cdot \xi_j \cdot x_{ij}^* + \hat{v}_i^{\xi} \cdot K_i \cdot y_i^*$$

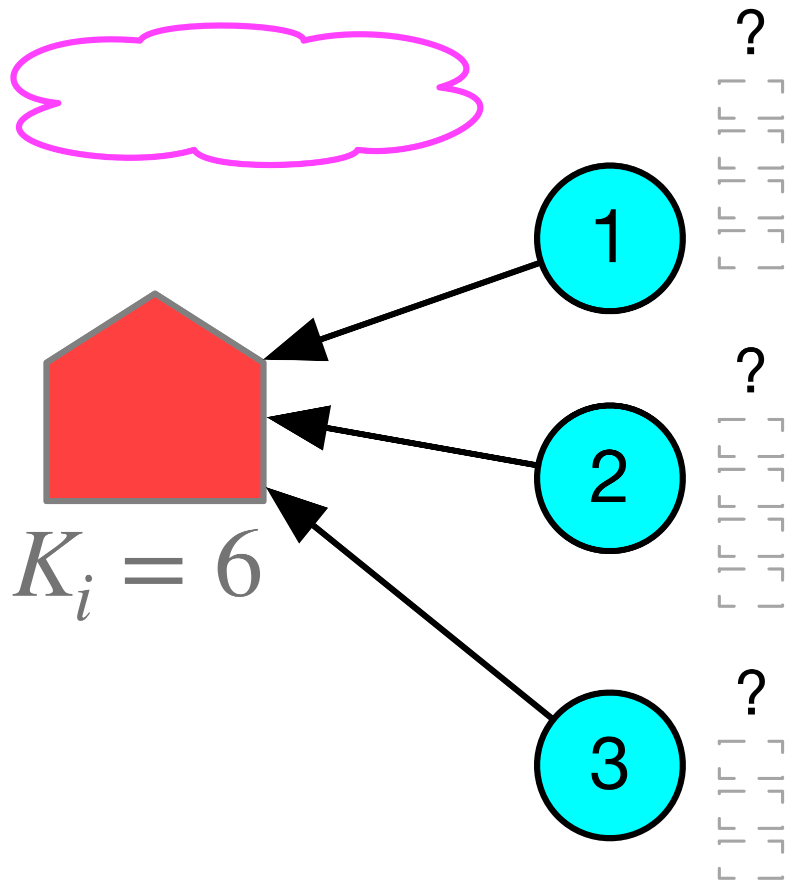
Stochastic facility location problems with outsourcing costs

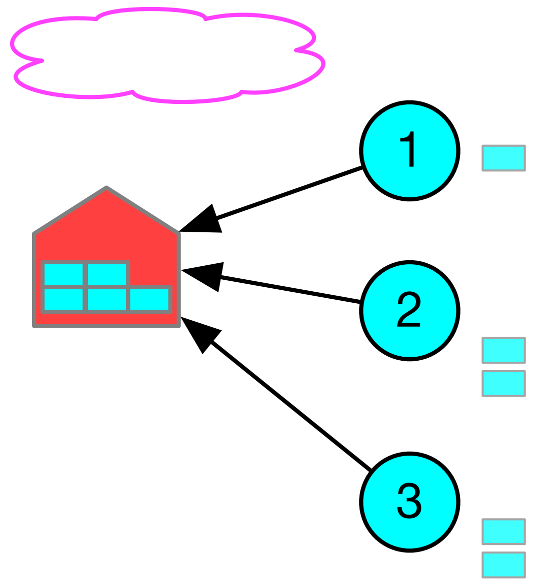
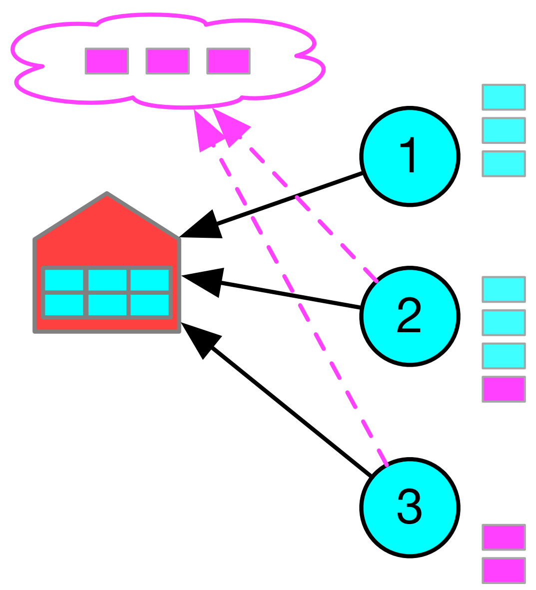
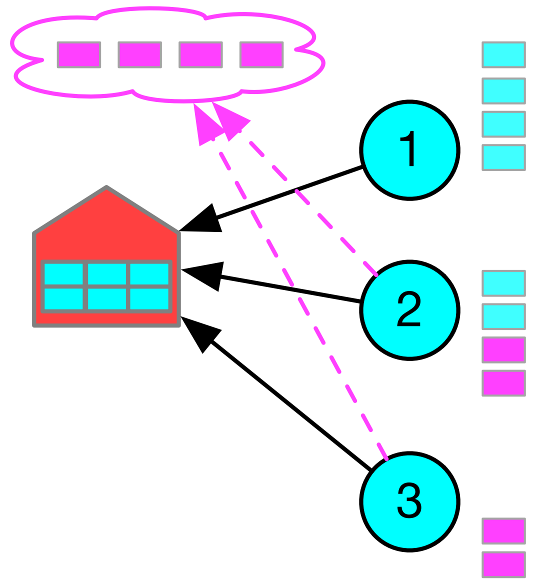
1. Bender formulation for a discrete set of scenarios (for example, a sample average approximation of the demand distributions)

2. **Bender formulation for general distributions.**

3. Computational experiments

Computing $\mathbb{E}\left[Q^i(x, y, \xi)\right]$

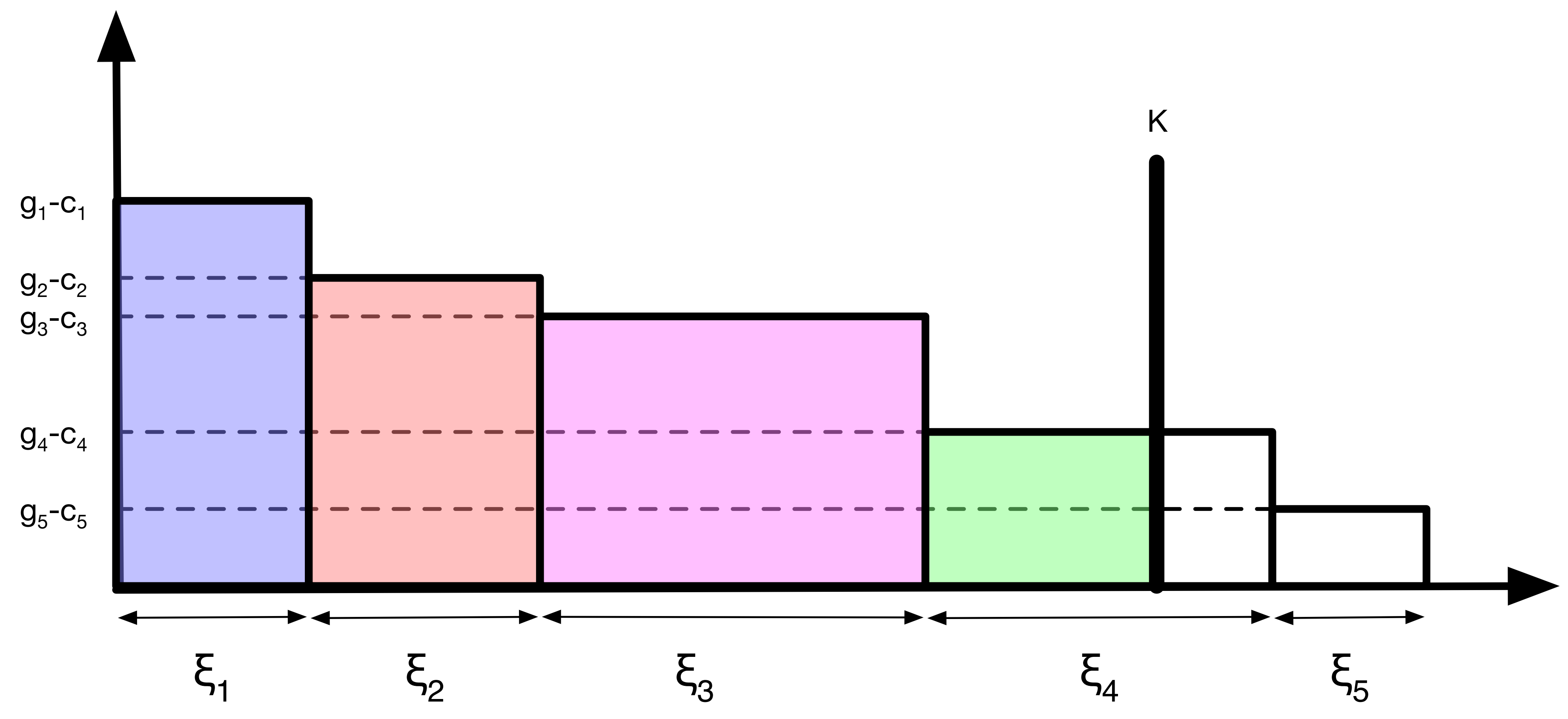


		Primal solution	Dual solution
Scenario 1		$w_{i1} = 1$ $w_{i2} = 2$ $w_{i3} = 2$	$\tau^i = \varnothing \Rightarrow \hat{v}_i = 0$ $\alpha_{i1} = g_{i1} - c_{i1}$ $\alpha_{i2} = g_{i2} - c_{i2}$ $\alpha_{i3} = g_{i3} - c_{i3}$
Scenario 2		$w_{i1} = 3$ $w_{i2} = 3$ $w_{i3} = 0$	$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$ $\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$ $\alpha_{i2} = 0$ $\alpha_{i3} = 0$
Scenario 3		$w_{i1} = 4$ $w_{i2} = 2$ $w_{i3} = 0$	$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$ $\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$ $\alpha_{i2} = 0$ $\alpha_{i3} = 0$

Expected value of the second-stage problem

$$\mathbb{E}\left[Q^i(x,y,\xi)\right] = \sum_{j\in J}(g_{ij} - c_{ij}) \cdot \mathbb{E}\left[w_{ij}^{\xi}\right]$$

Let $S_j^i(x,\xi)$ be the aggregated demand of the best j customers assigned to i: $S_j^i(x,\xi) = \sum_{l\leq j} \xi_l x_{il}$

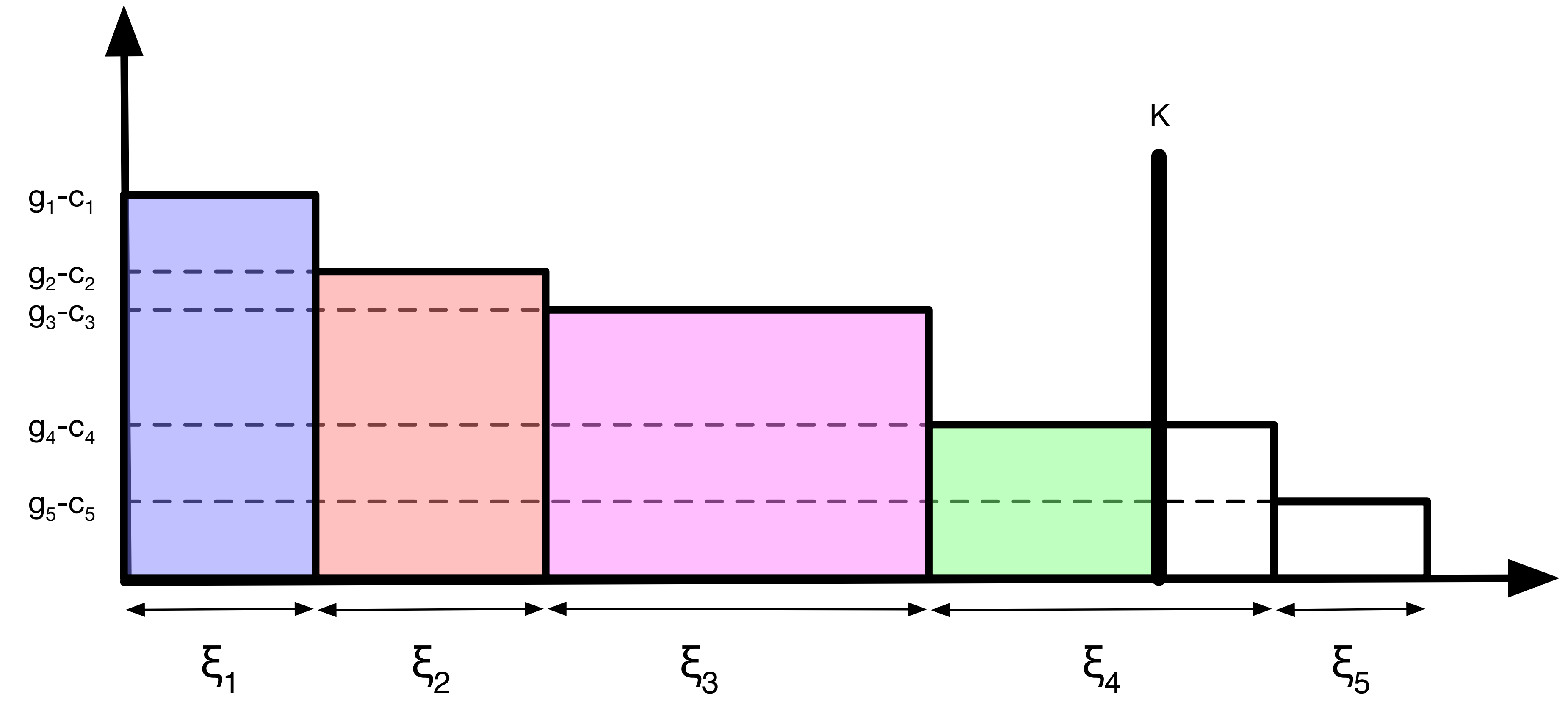


Expected value of the second-stage problem

$$\mathbb{E}\left[Q^i(x,y,\xi)\right] = \sum_{j\in J}(g_{ij}-c_{ij})\cdot\mathbb{E}\left[w_{ij}^{\xi}\right]$$

Let $S_j^i(x,\xi)$ be the aggregated demand of the best j customers assigned to i: $S_j^i(x,\xi) = \sum_{l\leq j}\xi_l x_{il}$

$$w_{ij}^{\xi} = \min\{S_j^i(x,\xi),K_i\} - \min\{S_{j-1}^i(x,\xi),K_i\}$$



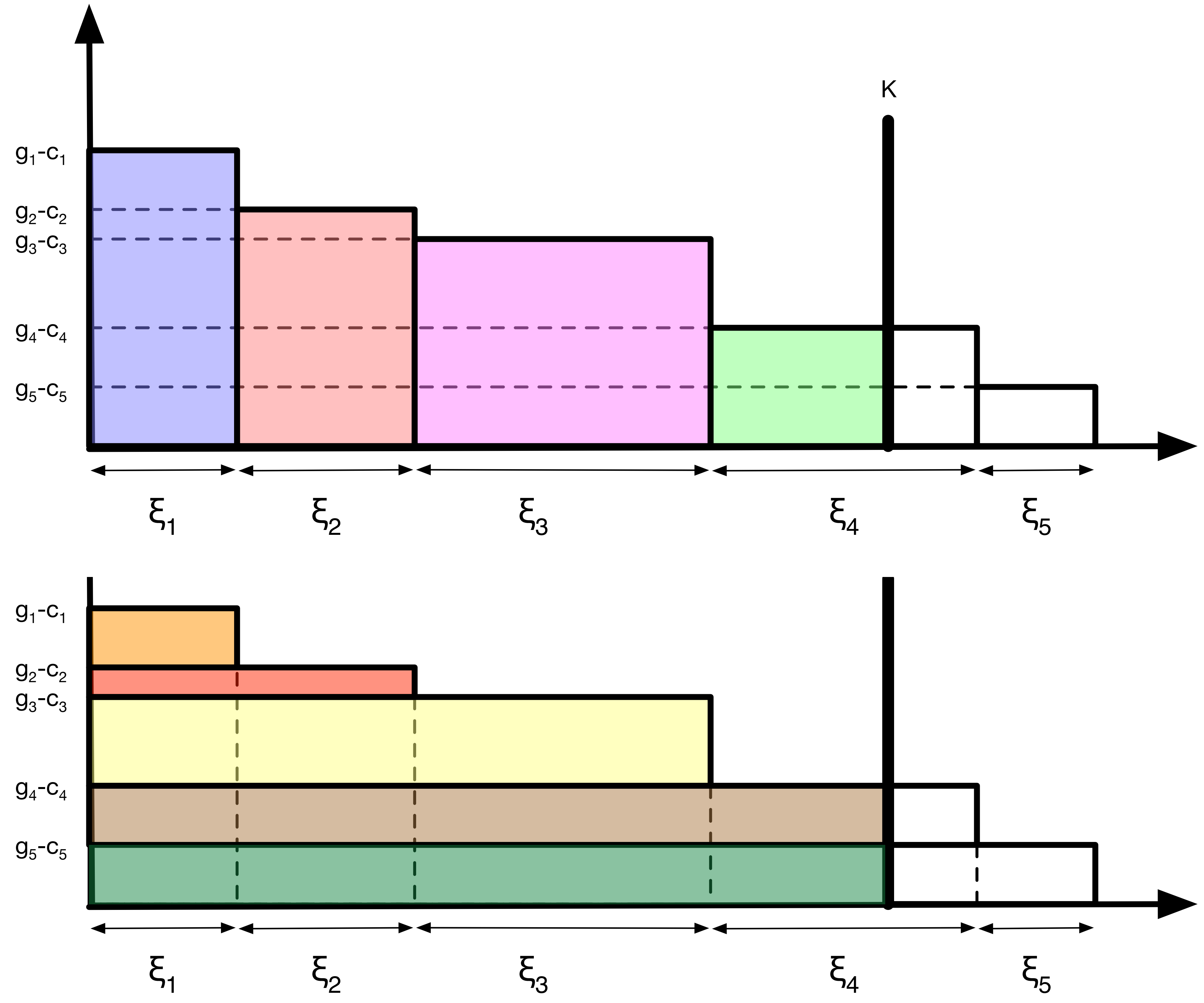
$$\mathbb{E}\left[Q^i(x,y,\xi)\right] = \sum_{j\in J}((g_{ij}-c_{ij})-(g_{i,j+1}-c_{i,j+1}))\cdot\mathbb{E}\left[\min\{S_j^i(x,\xi),K_i\}\right]$$

Expected value of the second-stage problem

$$\mathbb{E}\left[Q^i(x,y,\xi)\right] = \sum_{j\in J}(g_{ij}-c_{ij})\cdot\mathbb{E}\left[w^{\xi}_{ij}\right]$$

Let $S^i_j(x,\xi)$ be the aggregated demand of the best j customers assigned to i: $S^i_j(x,\xi) = \sum_{l\leq j}\xi_l x_{il}$

$$w^{\xi}_{ij} = \min\{S^i_j(x,\xi),K_i\} - \min\{S^i_{j-1}(x,\xi),K_i\}$$



$$\mathbb{E}\left[Q^i(x,y,\xi)\right] = \sum_{j\in J}((g_{ij}-c_{ij})-(g_{i,j+1}-c_{i,j+1}))\cdot\mathbb{E}\left[\min\{S^i_j(x,\xi),K_i\}\right]$$

Google Research

Expected value of the second-stage problem

For a fixed x we can obtain closed formulas for many demand distributions:

- Bernoulli distribution with mean μ_j: $\mathbb{E}[w_{ij}^{\xi}] = \mu_j \cdot F_{S_{j-1}}(K_i - 1)$
- Poisson distribution with mean μ_j:

$$\mathbb{E}[\min\{S_j(x,\xi), K_i\}] = K_i \cdot (1 - f_{Poisson(\mu_{S_j(x,\xi)})}(K_i)) + (\mu_{S_j(x,\xi)} - K_i) \cdot F_{Poisson(\mu_{S_j(x,\xi)})}(K_i - 1)$$

- Exponential distribution with mean μ:

$$\mathbb{E}[\min\{S_j(x,\xi), K_i\}] = j \cdot \mu \cdot F_{Gamma(j+1,1/\mu)}(K_i) + K_i \left(1 - F_{Gamma(j,1/\mu)}(K_i)\right)$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

Google Research

By the law of total probabilities:

$$\mathbb{E}\left[\min\{S_j^i(x,\xi),K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi),K_i\}\cdot 1_{S_j^i(x,\xi)\le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi),K_i\}\cdot 1_{S_j^i(x,\xi)>K_i}\right]$$

$$S_j^i(x,\xi) = \sum_{l\le j} \xi_l x_{il}$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

Google Research

By the law of total probabilities:

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi) \le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi) > K_i}\right]$$

$$= \mathbb{E}\left[S_j^i(x,\xi) \cdot 1_{S_j^i(x,\xi) \le K_i}\right] + \mathbb{E}\left[K_i \cdot 1_{S_j^i(x,\xi) > K_i}\right]$$

$$S_j^i(x,\xi) = \sum_{l \le j} \xi_l x_{il}$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

By the law of total probabilities:

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi)\le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi)> K_i}\right]$$

$$= \mathbb{E}\left[S_j^i(x,\xi) \cdot 1_{S_j^i(x,\xi)\le K_i}\right] + \mathbb{E}\left[K_i \cdot 1_{S_j^i(x,\xi)> K_i}\right]$$

$$= \sum_{l\le j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x,\xi)\le K_i}\right] \cdot x_{il} + K_i \cdot \mathbb{P}\left[S_j^i(x,\xi) > K_i\right]$$

$$S_j^i(x,\xi) = \sum_{l\le j} \xi_l x_{il}$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

By the law of total probabilities:

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi)\le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi)> K_i}\right]$$

$$= \mathbb{E}\left[S_j^i(x,\xi) \cdot 1_{S_j^i(x,\xi)\le K_i}\right] + \mathbb{E}\left[K_i \cdot 1_{S_j^i(x,\xi)> K_i}\right]$$

$$= \sum_{l\le j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x,\xi)\le K_i}\right] \cdot x_{il} + K_i \cdot \mathbb{P}\left[S_j^i(x,\xi) > K_i\right]$$

$$S_j^i(x,\xi) = \sum_{l\le j} \xi_l x_{il}$$

Non-linear function because coefficients also depends on x

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

Google Research

But we can use another first-stage assignment x' too

$$\mathbb{E}\left[\min\{S_j^i(x,\xi),K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi),K_i\}\cdot 1_{S_j^i(x',\xi)\le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi),K_i\}\cdot 1_{S_j^i(x',\xi)> K_i}\right]$$

$$\le \mathbb{E}\left[S_j^i(x,\xi)\cdot 1_{S_j^i(x',\xi)\le K_i}\right] + \mathbb{E}\left[K_i\cdot 1_{S_j^i(x',\xi)> K_i}\right]$$

$$= \sum_{l\le j}\mathbb{E}\left[\xi_l\cdot 1_{S_j^i(x',\xi)\le K_i}\right]\cdot x_{il} + K_i\cdot \mathbb{P}\left[S_j^i(x',\xi) > K_i\right]$$

Linear upper bound for a fixed x'

$$S_j^i(x,\xi) = \sum_{l\le j}\xi_l x_{il}$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

Lemma: $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$ is a concave function at x and

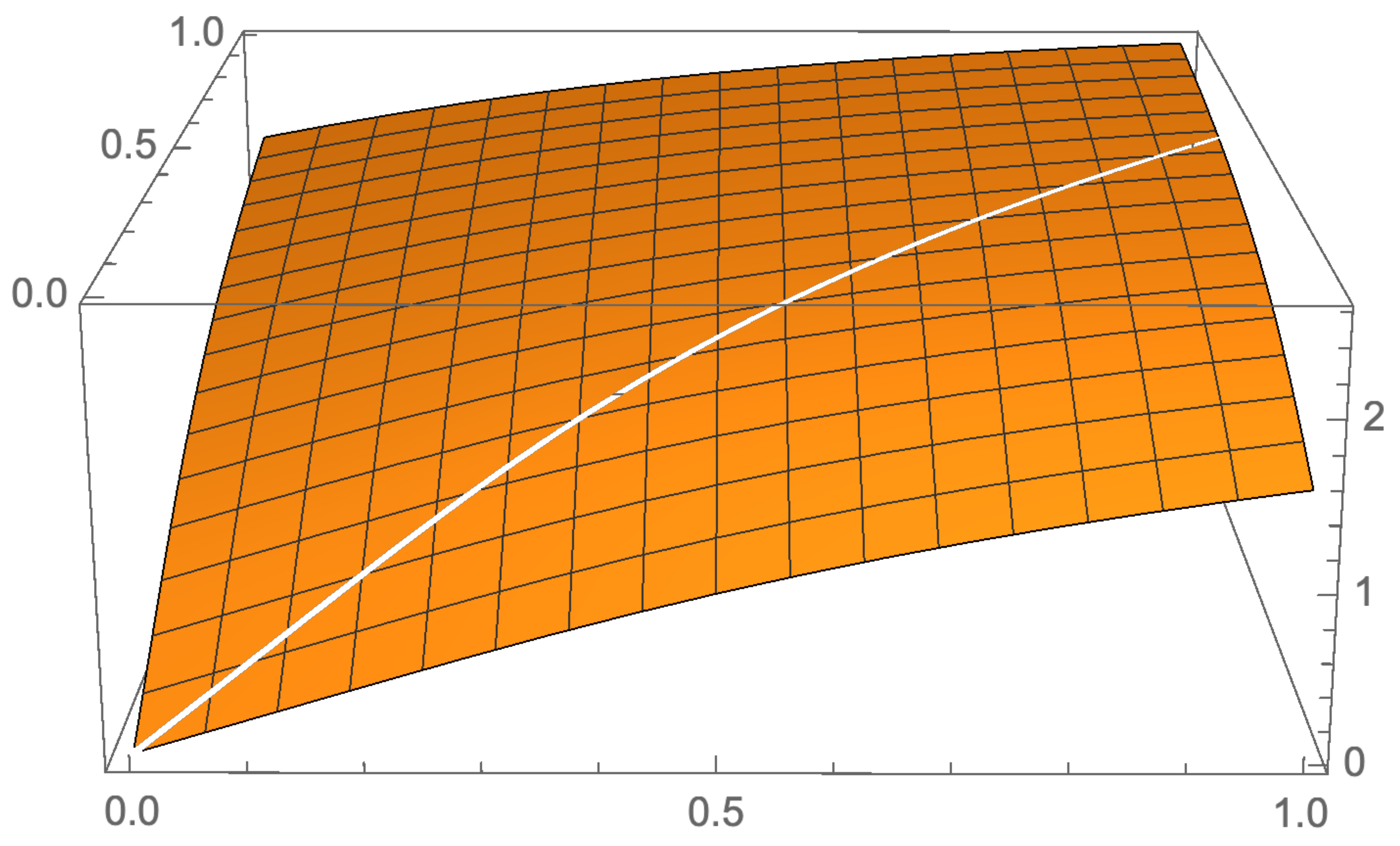
$$h_j^i(x,x') := \sum_{l\le j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x',\xi)\le K_i}\right] \cdot x_{il} + K_i \cdot \mathbb{P}\left[S_j^i(x',\xi) > K_i\right]$$

is in its subdifferential at x'

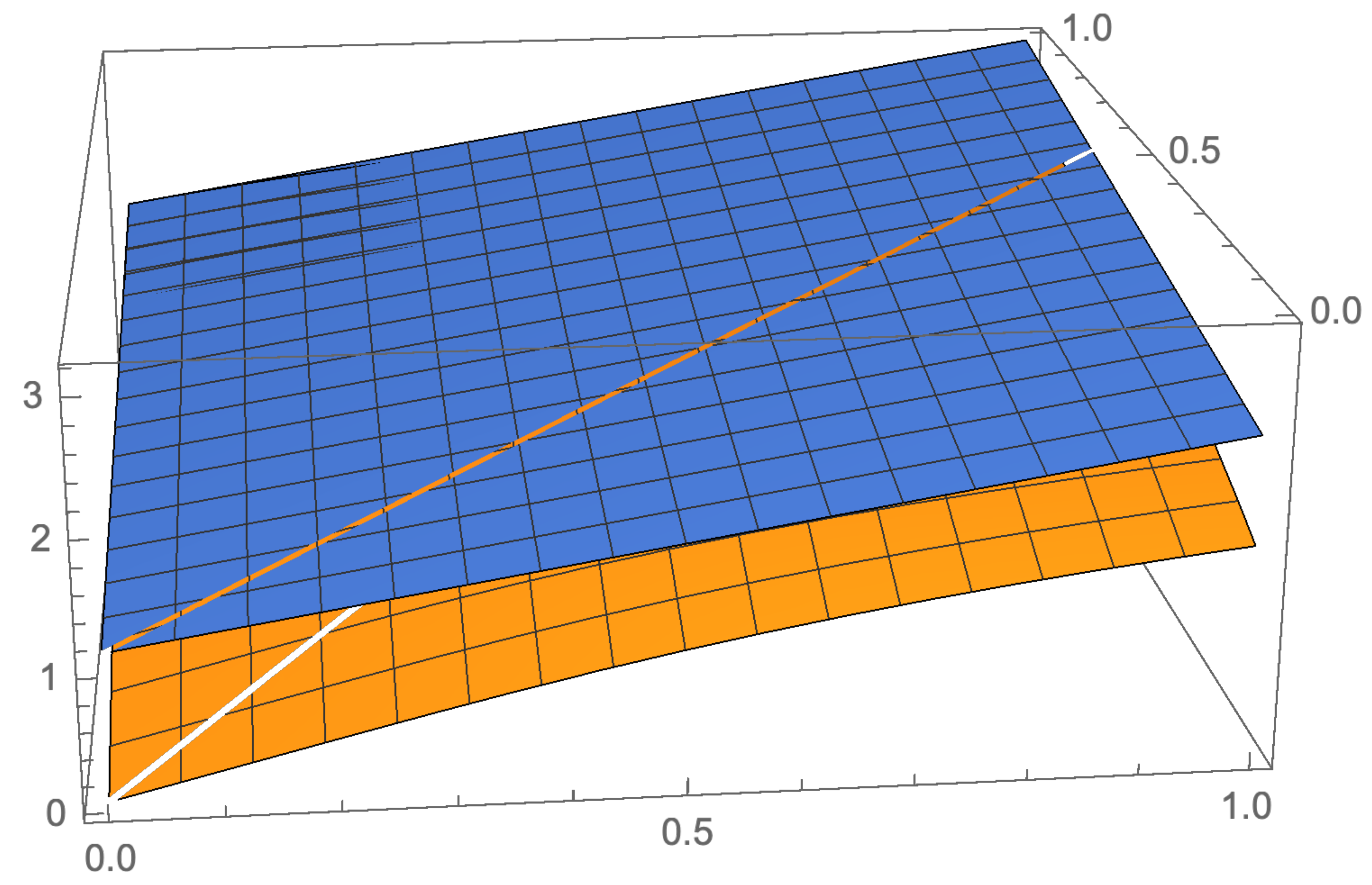
Google Research

Example:

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$ with K=3



$\mathbb{E}\left[\min\{S_2(x,\xi),3\}\right]$



$h_j^i(x, x')$ at $x' = (0,1)$

Google Research

Benders formulation for general distributions

We add a variable $z_{ij} \geq 0$ which correspond to the value of $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

$$\min_{x,y,z} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} g_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{j \in J} (c_{i,\sigma^i(j)} - c_{i,\sigma^i(j+1)}) z_{ij}$$

$$(x,y) \in \mathcal{X} \qquad x, y \in \{0,1\}$$

It can be solve by iteratively adding Generalized Benders optimality cuts for the given incumbent solution (x^*, y^*)

$$z_{ij} \leq \sum_{l \leq j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x',\xi) \leq K_i y_i'}\right] \cdot x_{il} + K_i \cdot y_i \cdot \mathbb{P}\left[S_j^i(x',\xi) > K_i y_i'\right]$$

Improving Benders formulation for general distributions

1) Submodularity of $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$. A set-valued function is sub modular if it has "diminishing returns".

- Lemma: Set-valued function $\mu(A) := \min\{S_j(1_A,\xi), K_i y_i\}$ is submodular for a given x, y, ξ.
- Corollary: Set-valued function $\mu'(A) := \mathbb{E}\left[\min\{S_j(1_A,\xi), K_i y_i\}\right]$ is sub-modular for a given x, y.

Improving Benders formulation for general distributions

Since $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$ is submodular, we can add submodular cuts (*)

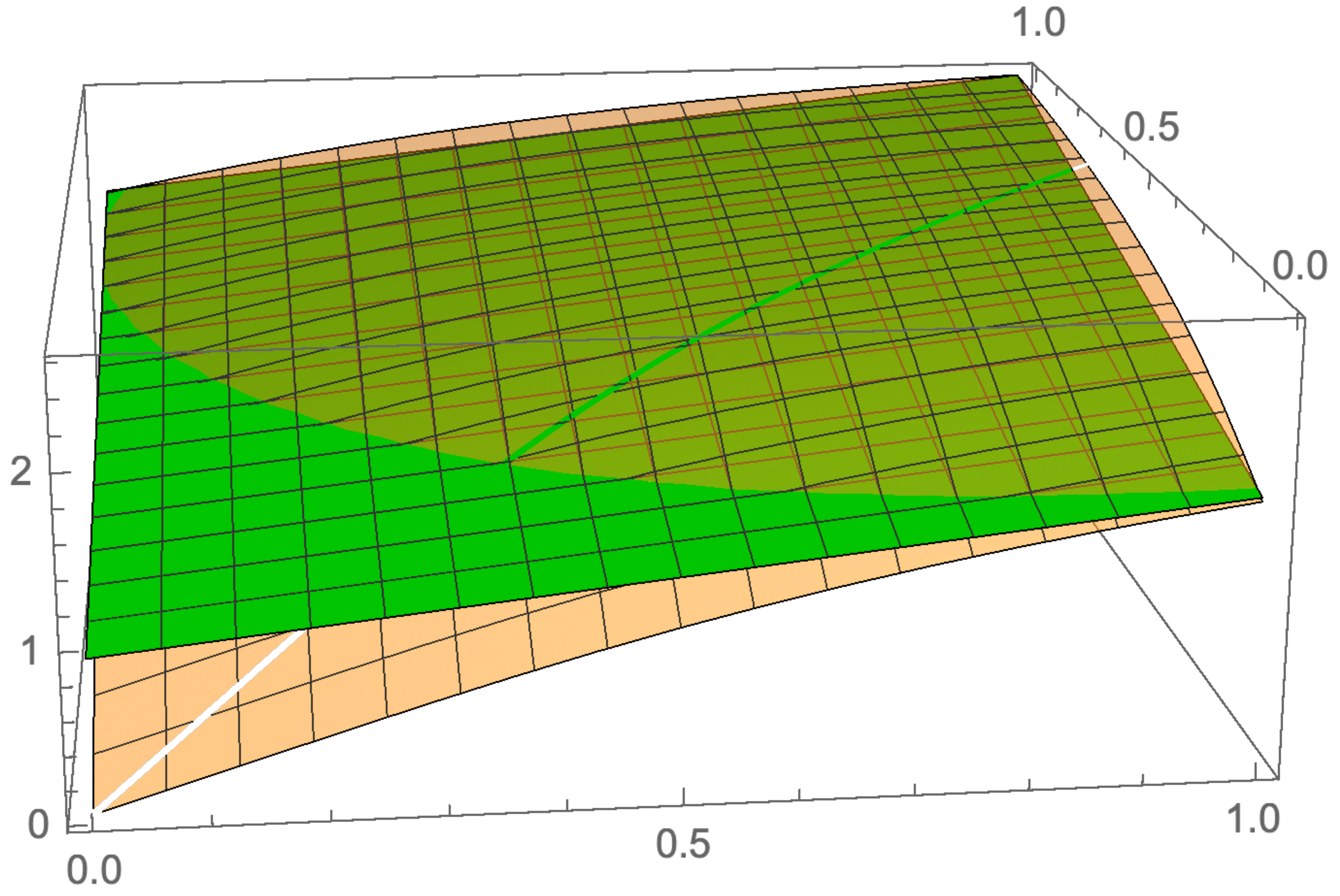
$$z_{ij} \leq \mathbb{E}[\min\{S_j(x'), K_i y_i\}] \cdot y_i + \sum_{l:x'_{il}=0} \mathbb{E}[\xi_l 1_{S_j(x')+\xi_l \leq K_i} + (K_i - S_j(x'))1_{S_j(x') \leq K_i < S_j(x')+\xi_l}] \cdot x_{il}$$

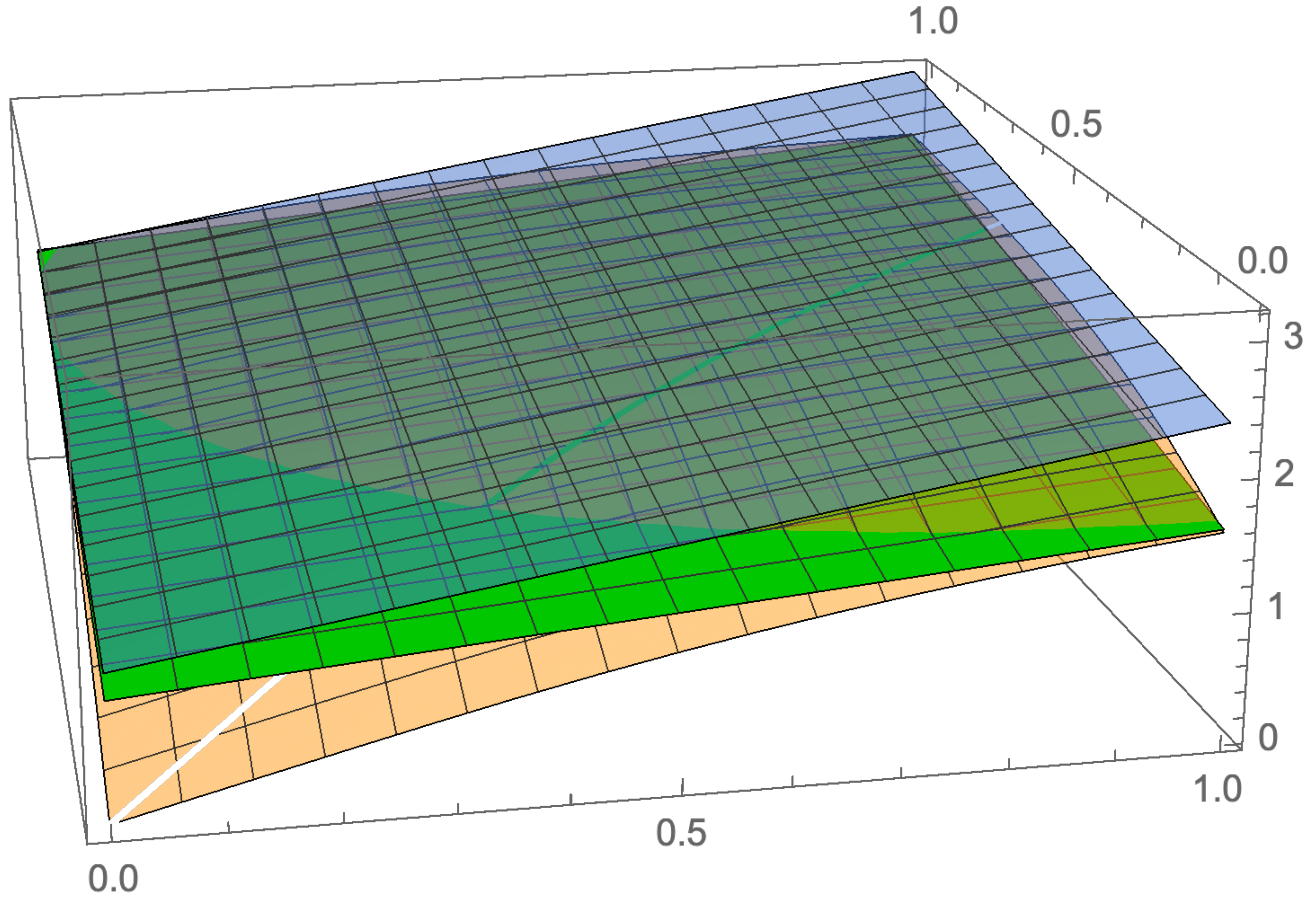
$$- \sum_{l:x'_{il}=1} \mathbb{E}[\xi_l 1_{\mathcal{S}_j \leq K_i} + (K_i - (\mathcal{S}_j - \xi_l))1_{\mathcal{S}_j - \xi_l \leq K_i < \mathcal{S}_j}] \cdot (1 - x_{il})$$

(*) see Nemhauser & Wolsey (1981), Ljubic & M. (2018)

Example:

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$ with K=3





Submodular cut at $x' = (0,1)$

Google Research

Improving Benders formulation for general distributions

2) Valid constraints on z_{ij} variables.

$$\left(z_{ij} \approx \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] \right)$$

$$z_{ij} \leq \sum_{l \in J : \sigma^i(l) \leq \sigma^i(j)} \mathbb{E}[\xi_l] x_{il}$$

$$z_{ij} \leq K_i y_i$$

$$z_{i,j-1} \leq z_{ij}$$

$$z_{ij} \leq z_{i,j-1} + \mathbb{E}[\xi_j] x_{ij}$$

Improving Benders formulation for general distributions

3) Both Bender and submodular cuts requires an integer solution x' to compute the coefficients to generate a cut. Can we create cuts for the relaxation of the problem?

If $\sum_{l \in J: \sigma^i(l) \leq \sigma^i(j)} x_{ij} = \kappa \in \mathbb{N}$, we can consider to sum the κ "worst" customers.

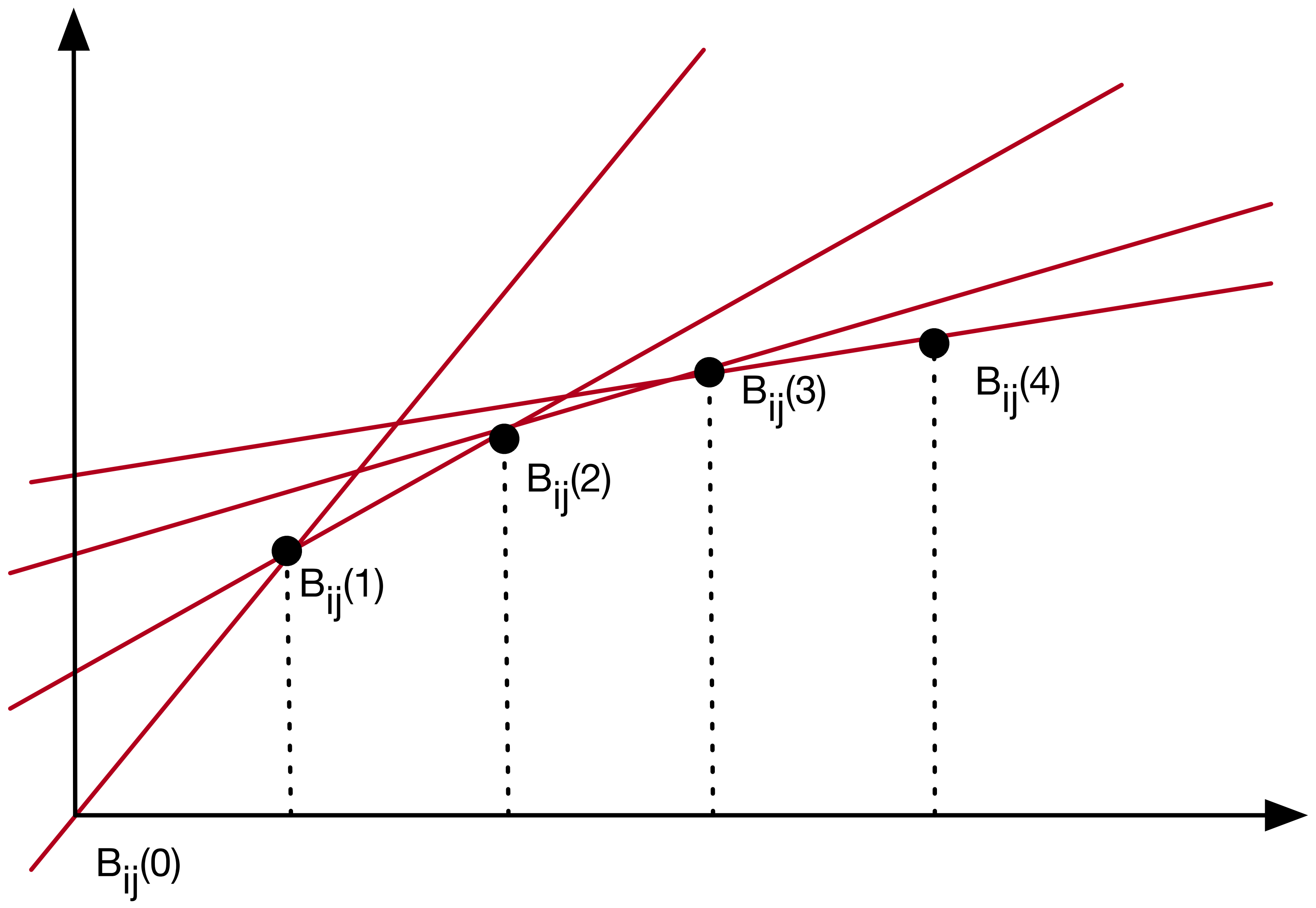
Proposition: Assume that random demands can be ordered in the usual stochastic order $\xi_{(1)} \geq_{st} \xi_{(2)} \geq_{st} \ldots \xi_{(j)}$. Then

$$\mathbb{E}\left[\min\{S_j(x), K_i y_i\}\right] \leq \mathbb{E}\left[\min\left\{\sum_{l=1}^{\kappa} \xi_{(l)}, K_i y_i\right\}\right]$$

Improving Benders formulation for general distributions

Let $B_{ij}(\kappa) := \mathbb{E}[\min\{\mathcal{S}_\kappa, K_i y_i\}]$ the expected value considering the "worst" κ customer. We can extend this function using a piece-wise linear function, creating the valid upper bound:

$$z_{ij} \leq B_{ij}(\kappa) + (B_{ij}(\kappa+1) - B_{ij}(\kappa)) \cdot \left(\sum_{l \in J : \sigma^i(l) \leq \sigma^i(j)} x_{il} - \kappa \right)$$



- For i.i.d. demand distributions, $\mathcal{S}_\kappa$ is the sum of κ random variables. The bound is tight.
- For single-parameter distribution (e.g. exponential or Poisson) the κ worst customers are the one with higher expected demand. The bound is not tight.

Google Research

Stochastic facility location problems with outsourcing costs

1. Bender formulation for a discrete set of scenarios (for example, a sample average approximation of the demand distributions)

2. Bender formulation for general distributions.

3. **Computational experiments**

Google Research

Computational experiments

Dataset for benchmarking:

- Albareda-Sambola et al (2011). 297 instances based on TSP problems, with 15 facilities and 30 customers.

- Random demands with Bernoulli and Exponential distributions with mean value 0.1, 0.5 or 0.9.

- Comparison of Benders, Submodular and PWL cuts using multi-cut (one cut for each z_{ij}) or a single-cut (aggregated cut for each facility).

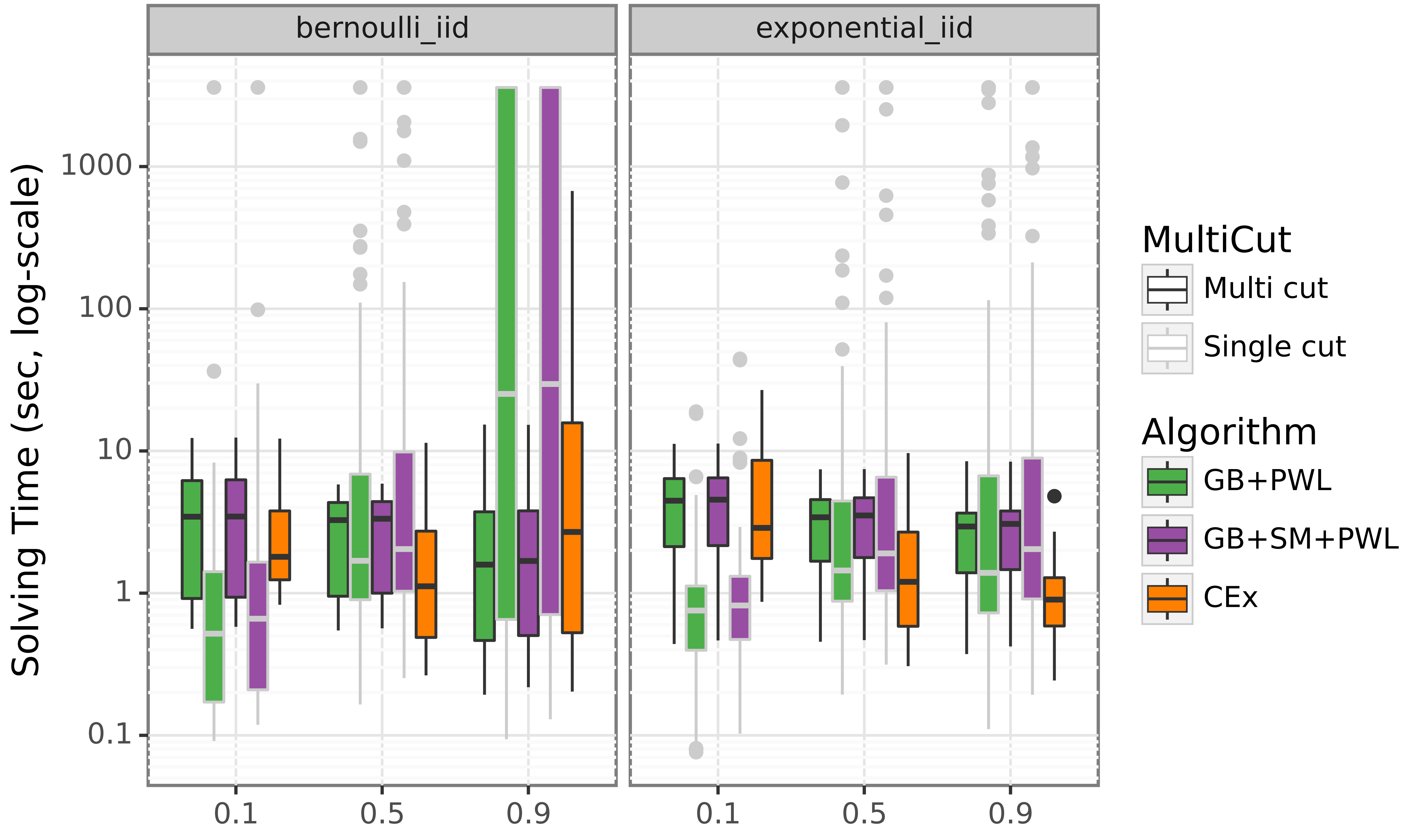
- Coded in C++ using Gurobi as solver.

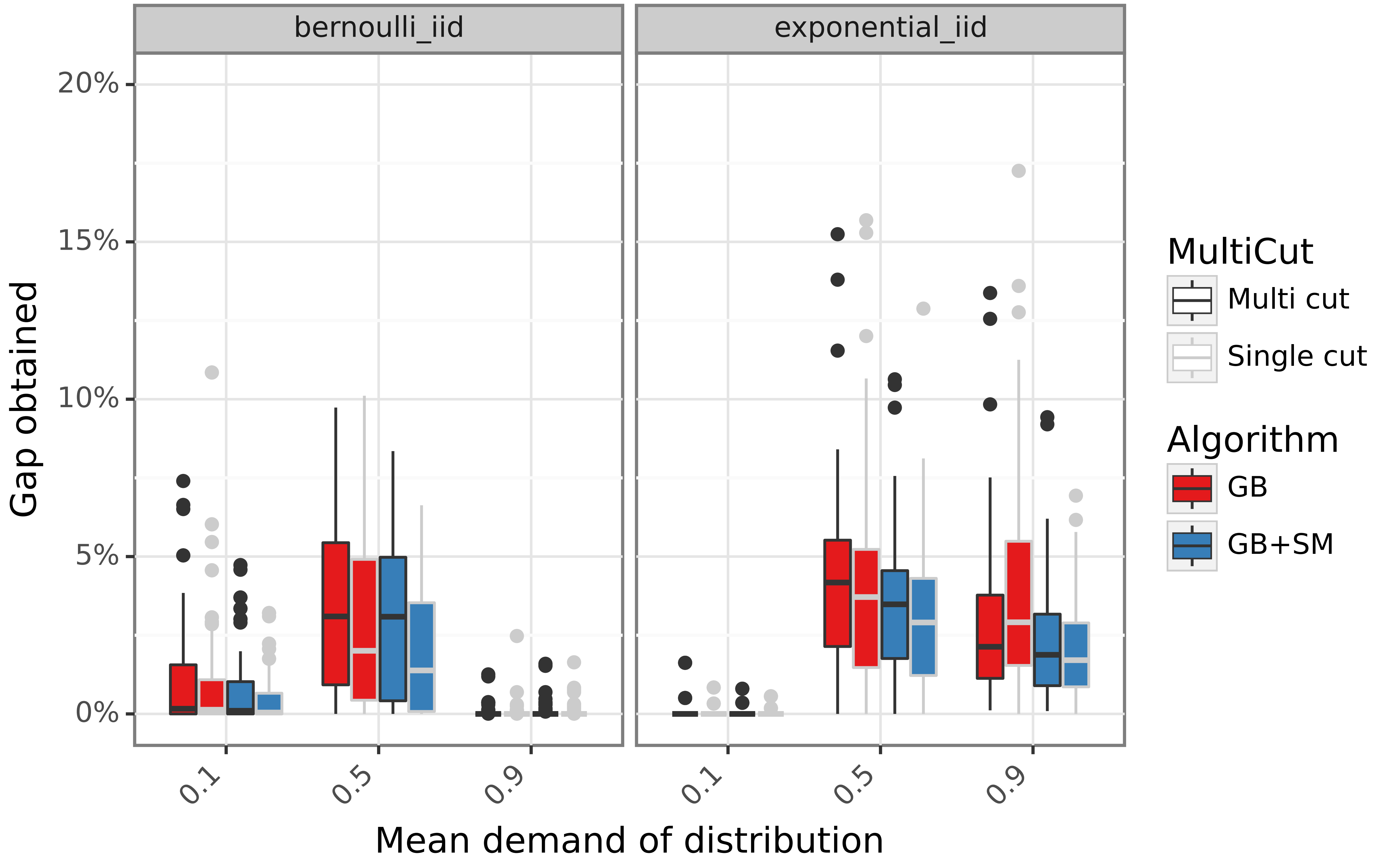
Performance profile for i.i.d. demands

- Adding PWL cuts solved all problems in a few seconds
- 50%/40% of instances are solved up to optimality. ~80% with <5% gap.

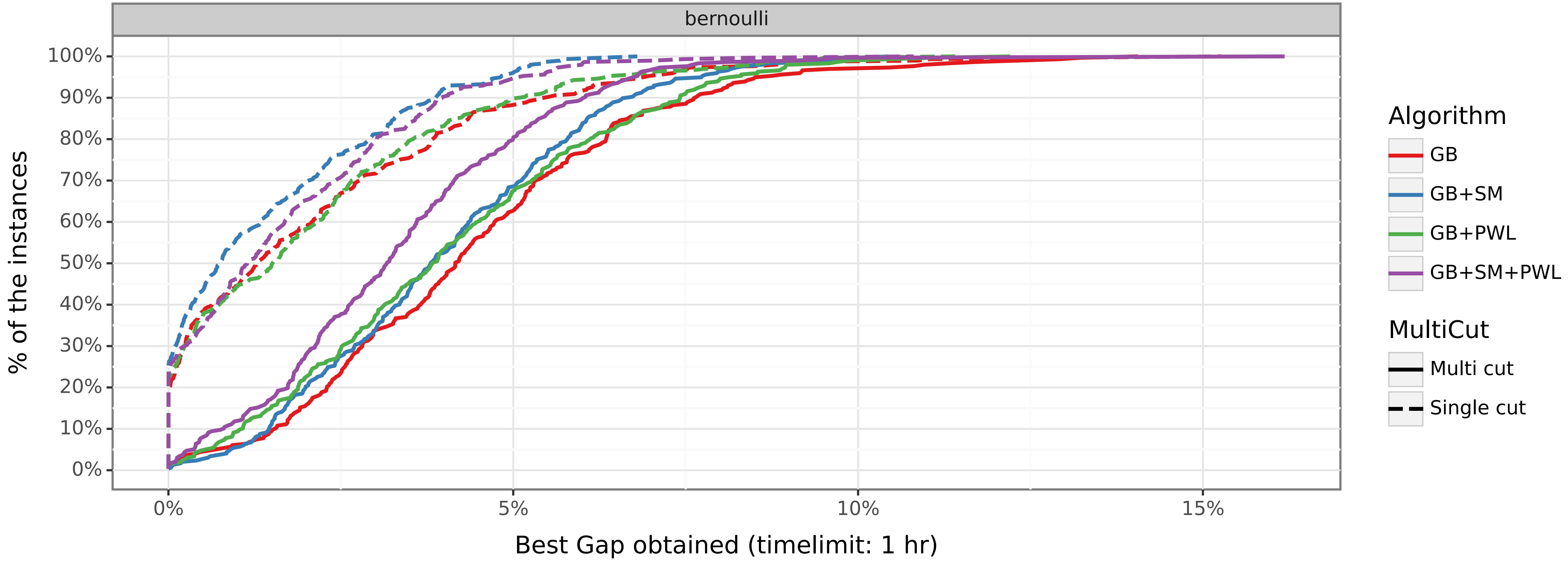
Google Research

Performance profile for i.i.d. demands



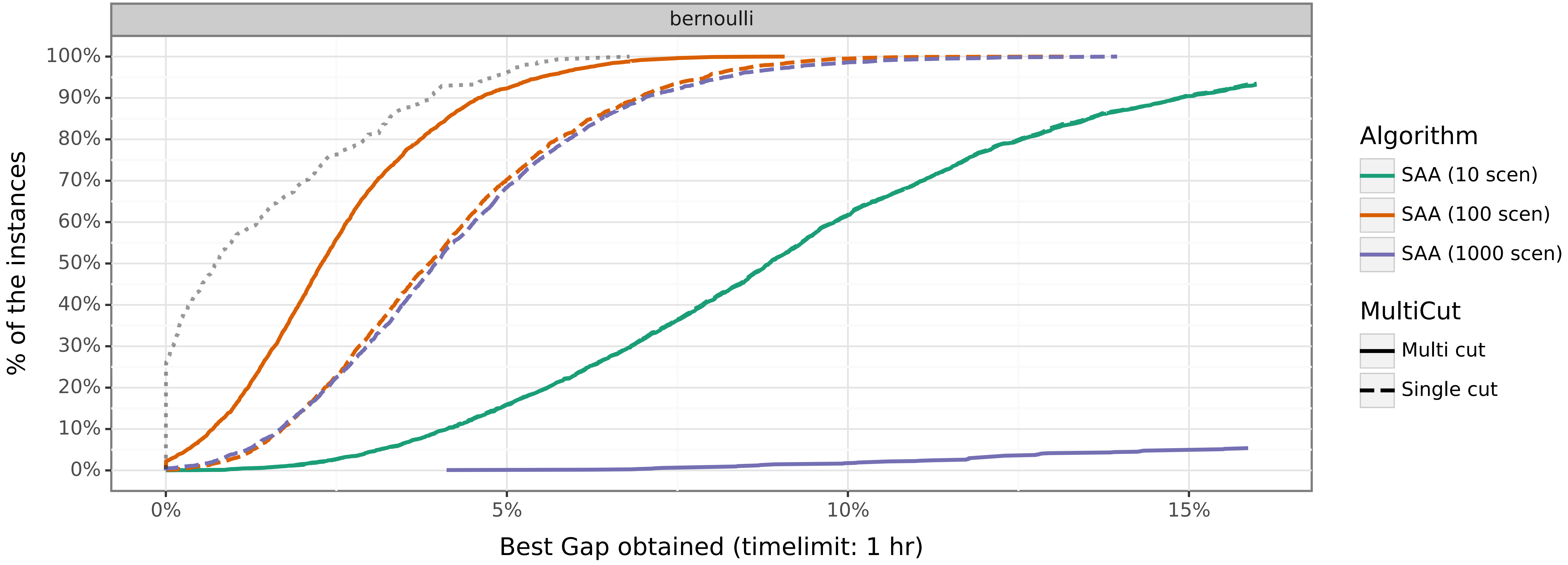


Performance profile for non-i.i.d. demands



- Too many cuts. Aggregating cuts performs better.
- PWL cuts are not longer efficient. Submodular cuts improve the performance
- Still > 90% of instances solved with <5% gap.

Comparison with sample average approximation



- SAA with 10 scenarios is solved up to optimality, but solution quality is very bad.
- Adding more scenarios improves the quality but became harder to solve.
- Generalized Bender outperforms SAA, particularly for smaller gaps.

Google Research

Conclusions

- Benders methodology for two-stage assignment problem where the second stage is a stochastic knapsack problem
  - An exact solution to the problem is achievable, precluding the necessity for scenario sampling.
  - We can exploit the structure of the subproblem: small number of optimal dual solutions where we can compute the expectations by conditioning.
  - Not an approximation! Provide true bounds for the problem.
  - Similar ideas can be extended to other problems.
  - See also Benders Adaptive Partition cuts (Ramirez-Pico & M., *Math Prog 2022*, Ramirez-Pico, Ljubic, M., *Transp Sci 2023*).

Google Research

Compact formulation for i.i.d. distributions

Case: customer demands are i.i.d.: cumulative demand S_l of the l ~~best~~ customers doesn't depend on which customer are the best.

Let $C_{il} := \mathbb{E}\left[\max\left\{S_l, K_i\right\}\right] - \mathbb{E}\left[\max\left\{S_{l-1}, K_i\right\}\right]$ (Can be precomputed using previous formulas)

then $$\mathbb{E}[Q^i(x, y, \xi)] = \sum_{j \in J} (g_{ij} - c_{ij}) \cdot C_{i, r^i_{\hat{x}}(j)} \cdot x_{ij}$$

where $r^i_x(j)$ is then is the *ranking* of customer j among the customers assigned to facility i in decreasing order of $(g_{ij} - c_{ij})$

Compact formulation for i.i.d. distributions

New variable: $z_{ijl} \in \{0,1\}$ if customer j is the l-best customer assigned to i

$$\min \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} g_{ij} \mu x_{ij} - \sum_{i \in I} \sum_{j \in J} (g_{ij} - c_{ij}) \cdot \mathcal{C}_{il} \cdot z_{ijl}$$

$$\sum_{i} x_{ij} = 1 \qquad x_{ij} \leq y_i \qquad Ax + By \leq h$$

Constraints from main problem

$$x_{ij} = \sum_{l=1}^{|J|} z_{ijl}$$

If assigned then it must have a ranking position

$$\sum_{j \in J} z_{ijl} \leq y_i$$

At most 1 customer on each ranking position

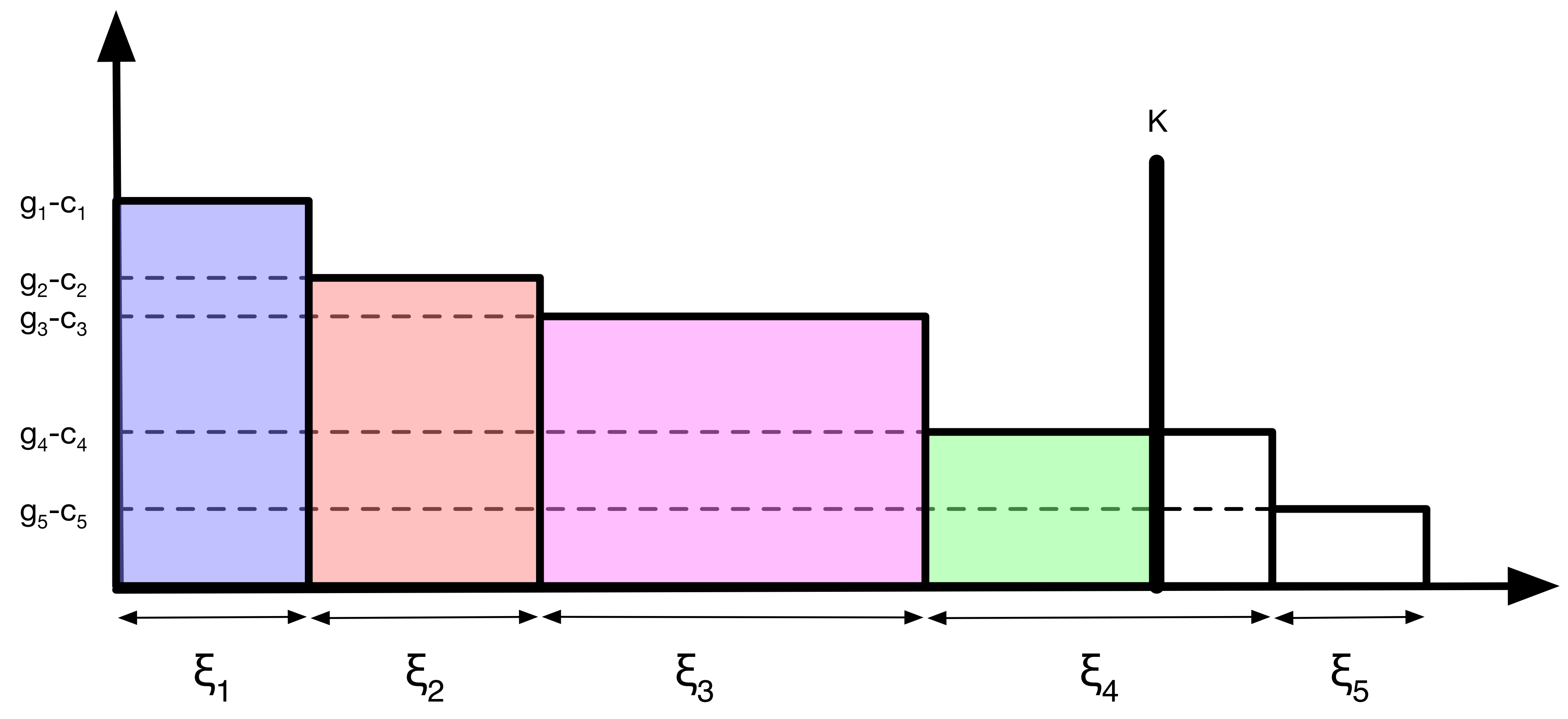
$$\sum_{l=1}^{|J|} (l-1) \cdot z_{ijl} \leq \sum_{k: g_{ik} - c_{ik} > g_{ij} - c_{ij}} x_{ik}$$

If $z_{ijl} = 1$ then at least $l - 1$ other customers with higher profit must be assigned to i

Expected value of the second-stage problem

$$\mathbb{E}\left[Q^i(x,y,\xi)\right] = \sum_{j\in J} (g_{ij} - c_{ij}) \cdot \mathbb{E}\left[w_{ij}^{\xi}\right]$$

Let $S_j^i(x,\xi)$ be the aggregated demand of the best j customers assigned to i: $S_j^i(x,\xi) = \sum_{l\leq j} \xi_l x_{il}$



Then:

$$\mathbb{E}\left[w_{ij}^{\xi}\right] := \mathbb{E}\left[\xi_j \cdot 1_{S_j^i(x,\xi)\leq K_i}\right] + \mathbb{E}\left[\left(K_i - S_{j-1}^i(x,\xi)\right)^+ \cdot 1_{S_j^i(x,\xi)>K_i}\right]$$

Benders formulation for general distributions.

- Previous formulation only applies to a finite set of scenarios (for example, a Bernoulli distribution, or sampled scenarios from the original distribution).
- But we can explote the structure of the dual subproblem to solve this problem for general distributions.

$$\min_{x,y} \sum_{i\in I} f_i y_i + \sum_{i\in I} \sum_{j\in J} g_{ij}\mu_j x_{ij} - \sum_{i\in I} \Theta_i$$

$$x_{ij} \le y_i \qquad Ax + By \le h \qquad x, y \in \{0,1\}$$

$$\Theta_i \le \mathbb{E}\left[Q^i(x, y, \xi)\right]$$

Computing $\mathbb{E}\left[Q^i(x, y, \xi)\right]$

We know how to solve the dual subproblem for a given scenario

$$Q^i(x, y, \xi) = \sum_{j \in J} \left(g_{ij} - c_{ij} - \hat{v}_i^{\xi}\right)^+ \cdot \xi_j \cdot x_{ij} + \hat{v}_i^{\xi} \cdot K_i \cdot y_i$$

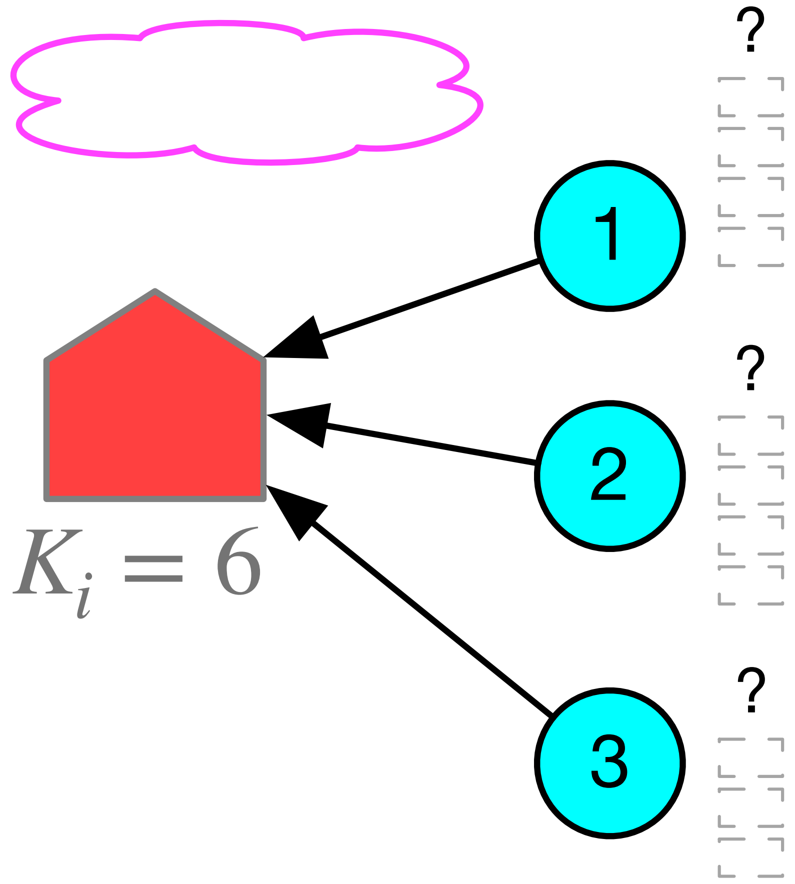
but we need to compute their expected value over all scenarios.

In the (unlikely!) case that the dual solution $\hat{v}_i^{\xi}$ is the same for all scenarios, then

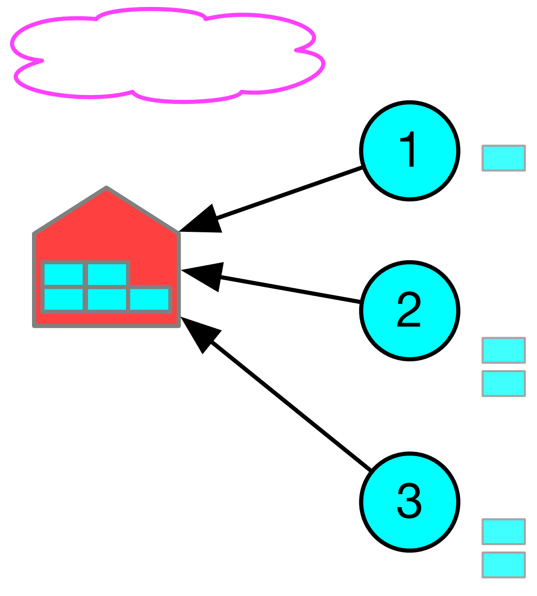
$$\mathbb{E}[Q^i(x, y, \xi)] = \sum_{j \in J} \left(g_{ij} - c_{ij} - \hat{v}_i^{\xi}\right)^+ \cdot \mathbb{E}[\xi_j] \cdot x_{ij} + \hat{v}_i^{\xi} \cdot K_i \cdot y_i$$

Unlikely, but….

Computing $\mathbb{E}\left[Q^i(x, y, \xi)\right]$



Scenario 1



Primal solution

$w_{i1} = 1$

$w_{i2} = 2$

$w_{i3} = 2$

Dual solution

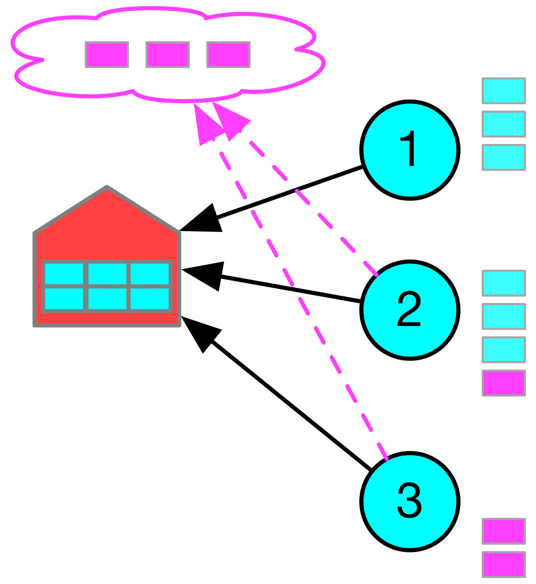
$\tau^i = \varnothing \Rightarrow \hat{v}_i = 0$

$\alpha_{i1} = g_{i1} - c_{i1}$

$\alpha_{i2} = g_{i2} - c_{i2}$

$\alpha_{i3} = g_{i3} - c_{i3}$

Scenario 2



$w_{i1} = 3$

$w_{i2} = 3$

$w_{i3} = 0$

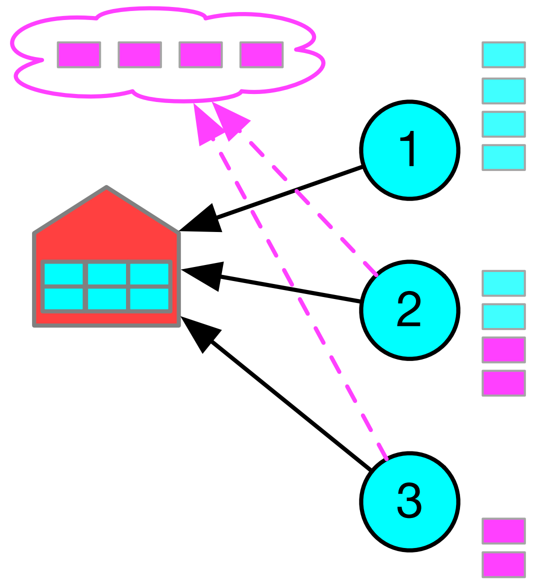
$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$

$\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$

$\alpha_{i2} = 0$

$\alpha_{i3} = 0$

Scenario 3



$w_{i1} = 4$

$w_{i2} = 2$

$w_{i3} = 0$

$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$

$\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$

$\alpha_{i2} = 0$

$\alpha_{i3} = 0$

Computing $\mathbb{E}\left[Q^i(x, y, \xi)\right]$

Let $P = \{\xi : \tau^i(\xi) = k\}$ be the set of scenarios where customer k is the critical one. Then

$$\mathbb{E}[Q^i(x, y, \xi) \mid P] = \mathbb{E}\left[\sum_{j \in J} \left(g_{ij} - c_{ij} - \hat{v}_i^\xi\right)^+ \cdot \xi_j \cdot x_{ij} + \hat{v}_i^\xi \cdot K_i \cdot y_i \mid P\right]$$

$$= \sum_{j \in J} \left(g_{ij} - c_{ij} - \hat{v}_i^\xi\right)^+ \cdot \mathbb{E}[\xi_j \mid P] \cdot x_{ij} + \hat{v}_i^\xi \cdot K_i \cdot y_i$$

Google Research

Computing $\mathbb{E}\left[Q^i(x, y, \xi)\right]$

Let $P = \{\xi : \tau^i(\xi) = k\}$ be the set of scenarios where customer k is the critical one. Then

$$\mathbb{E}[Q^i(x,y,\xi) \mid P] = \mathbb{E}\left[\sum_{j\in J}\left(g_{ij} - c_{ij} - \hat{v}_i^{\xi}\right)^+ \cdot \xi_j \cdot x_{ij} + \hat{v}_i^{\xi} \cdot K_i \cdot y_i \mid P\right]$$

$$= \sum_{j\in J}\left(g_{ij} - c_{ij} - \hat{v}_i^{\xi}\right)^+ \cdot \mathbb{E}[\xi_j \mid P] \cdot x_{ij} + \hat{v}_i^{\xi} \cdot K_i \cdot y_i$$

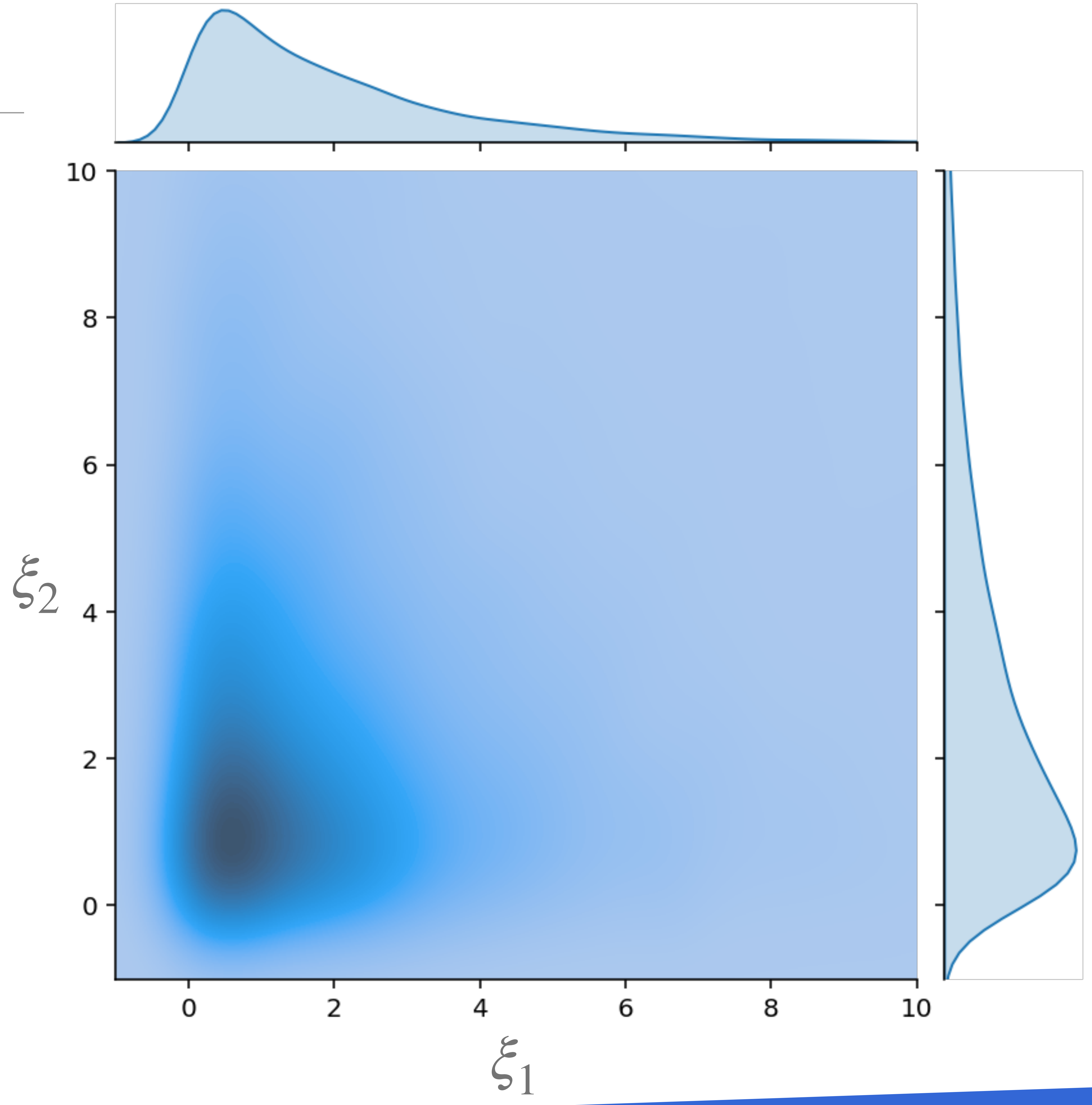
And by the law of total probabilities, for any partition $\mathscr{P}$ of the probability space:

$$\mathbb{E}[Q^i(x,y,\xi)] = \sum_{P\in\mathscr{P}} \mathbb{E}[Q^i(x,y,\xi) \mid P] \cdot \mathbb{P}[P]$$

Hence, we can partition the probability space into at most 1 + # customers assigned to the facility subsets.

Example TO FIX

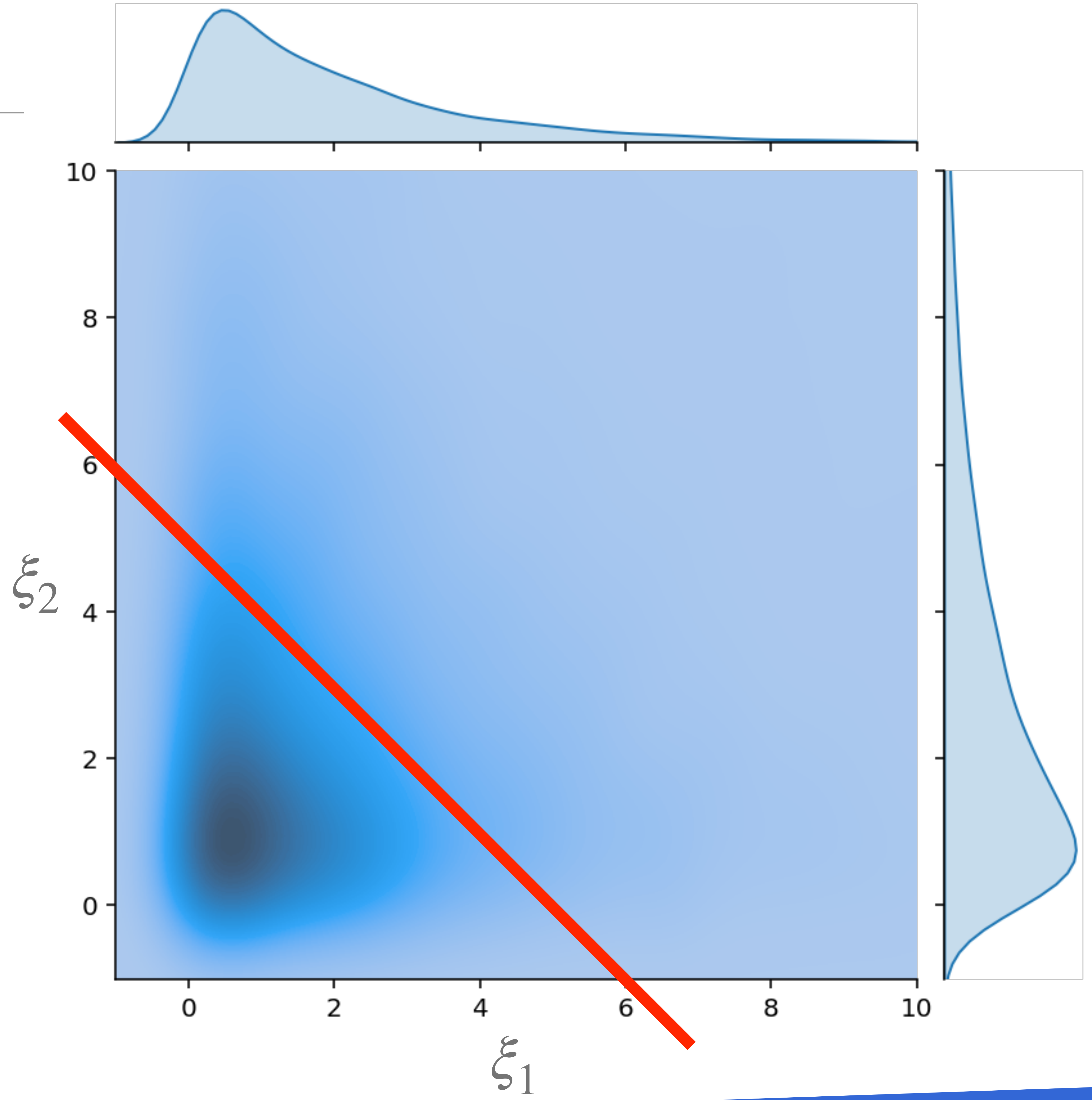
Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$.



Example TO FIX

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$.

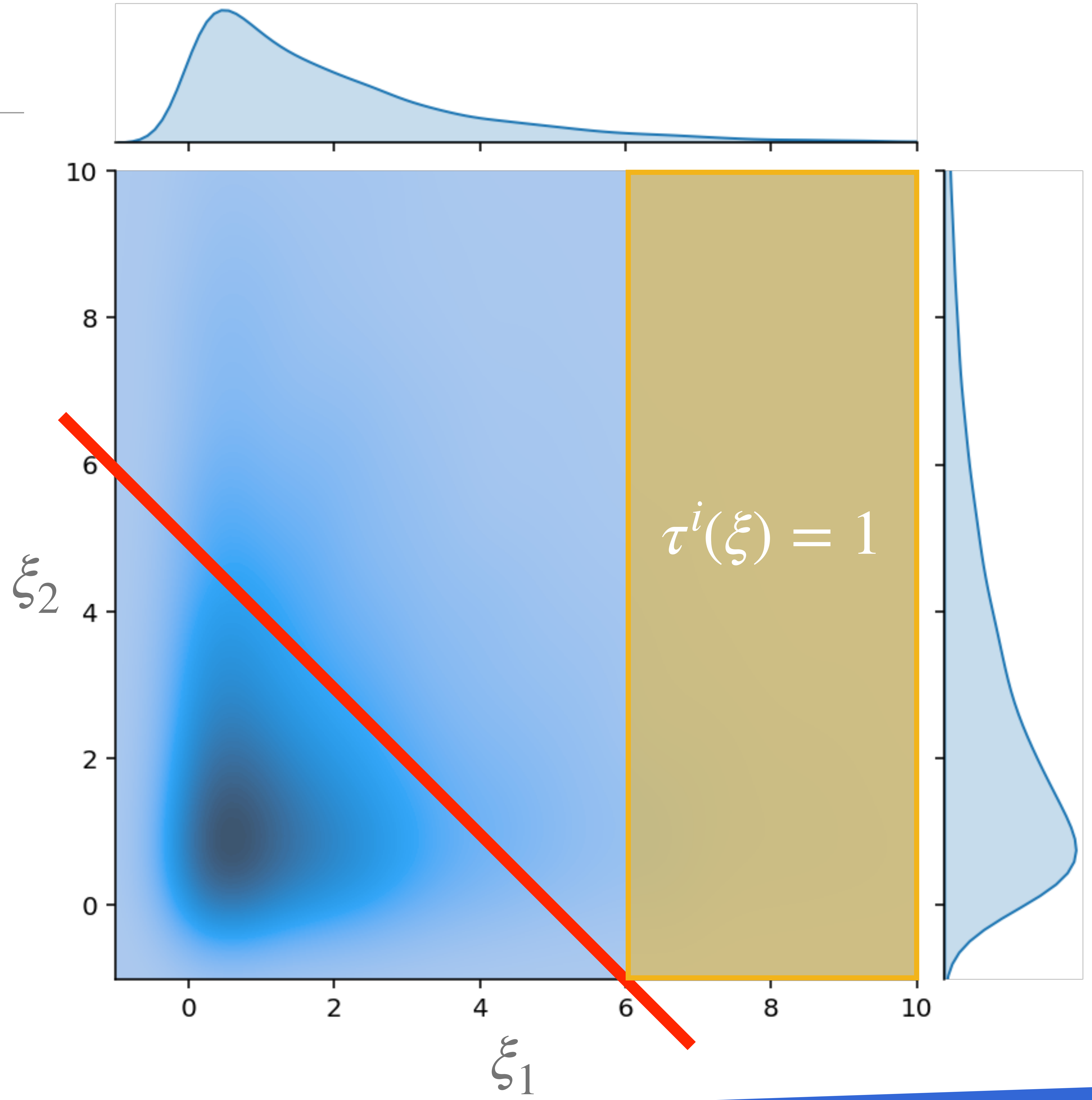
Capacity $K^i = 6$



Example TO FIX

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$.

Capacity $K^i = 6$

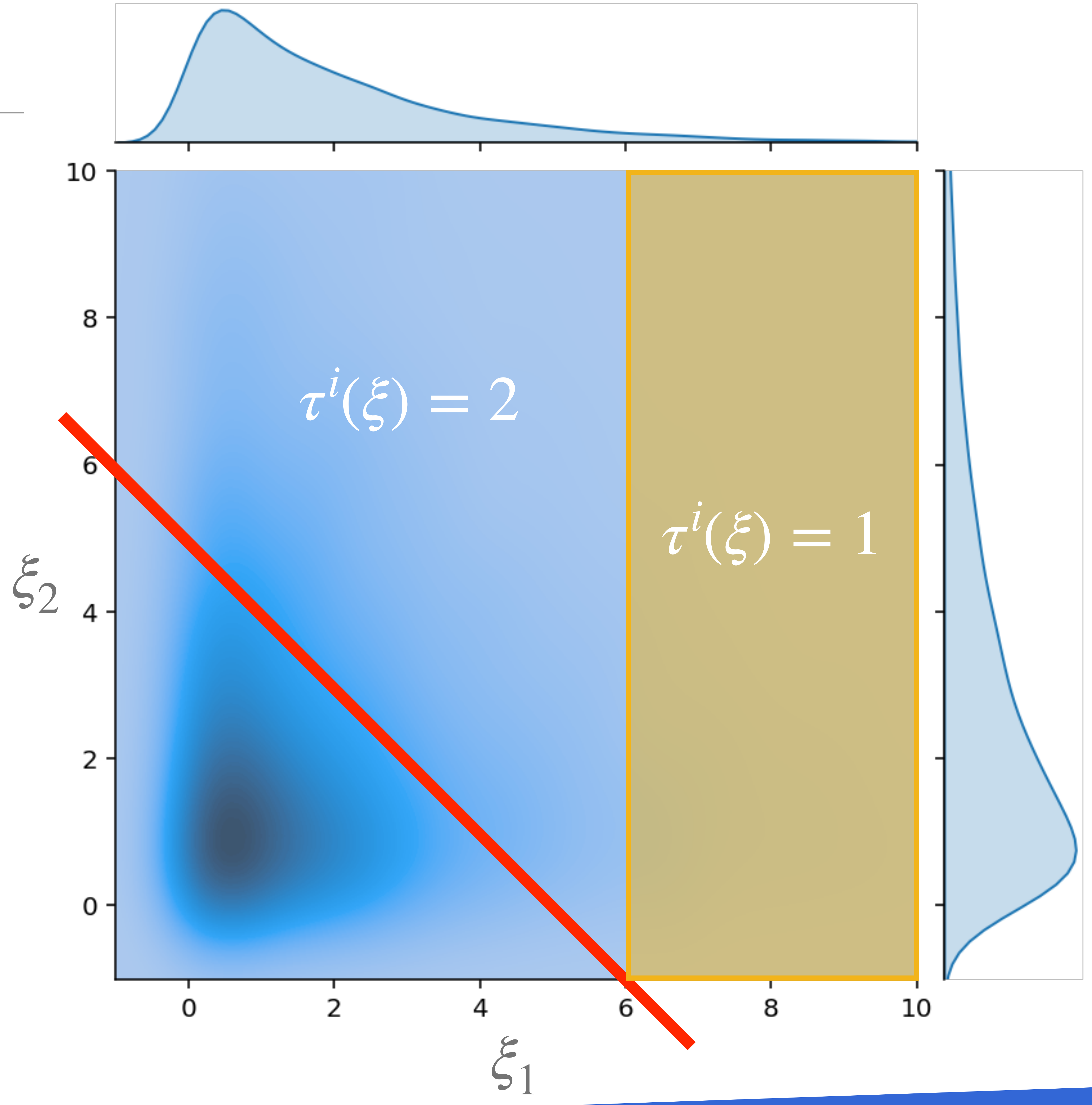


Google Research

Example TO FIX

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$.

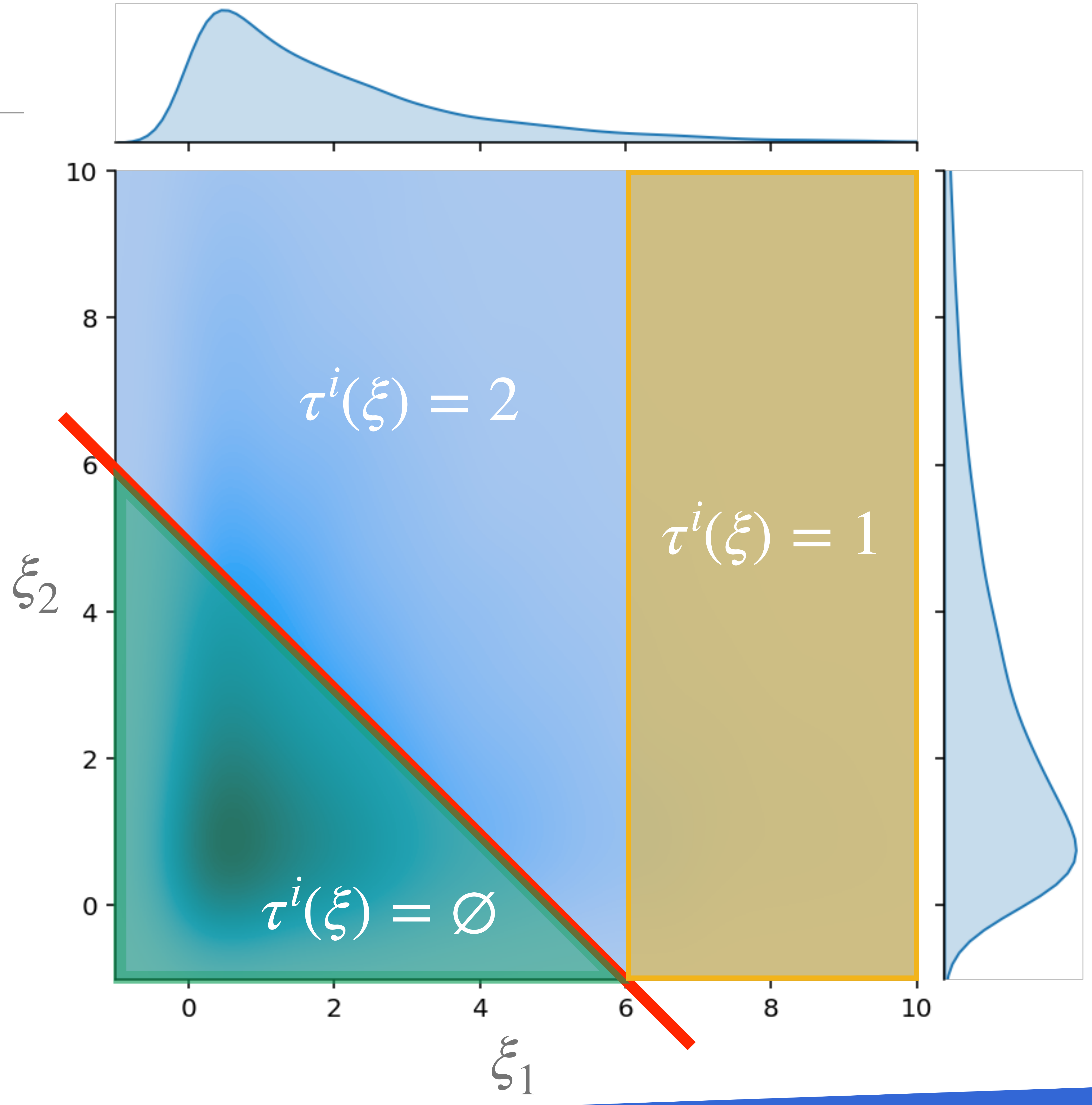
Capacity $K^i = 6$



Example TO FIX

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$.

Capacity $K^i = 6$

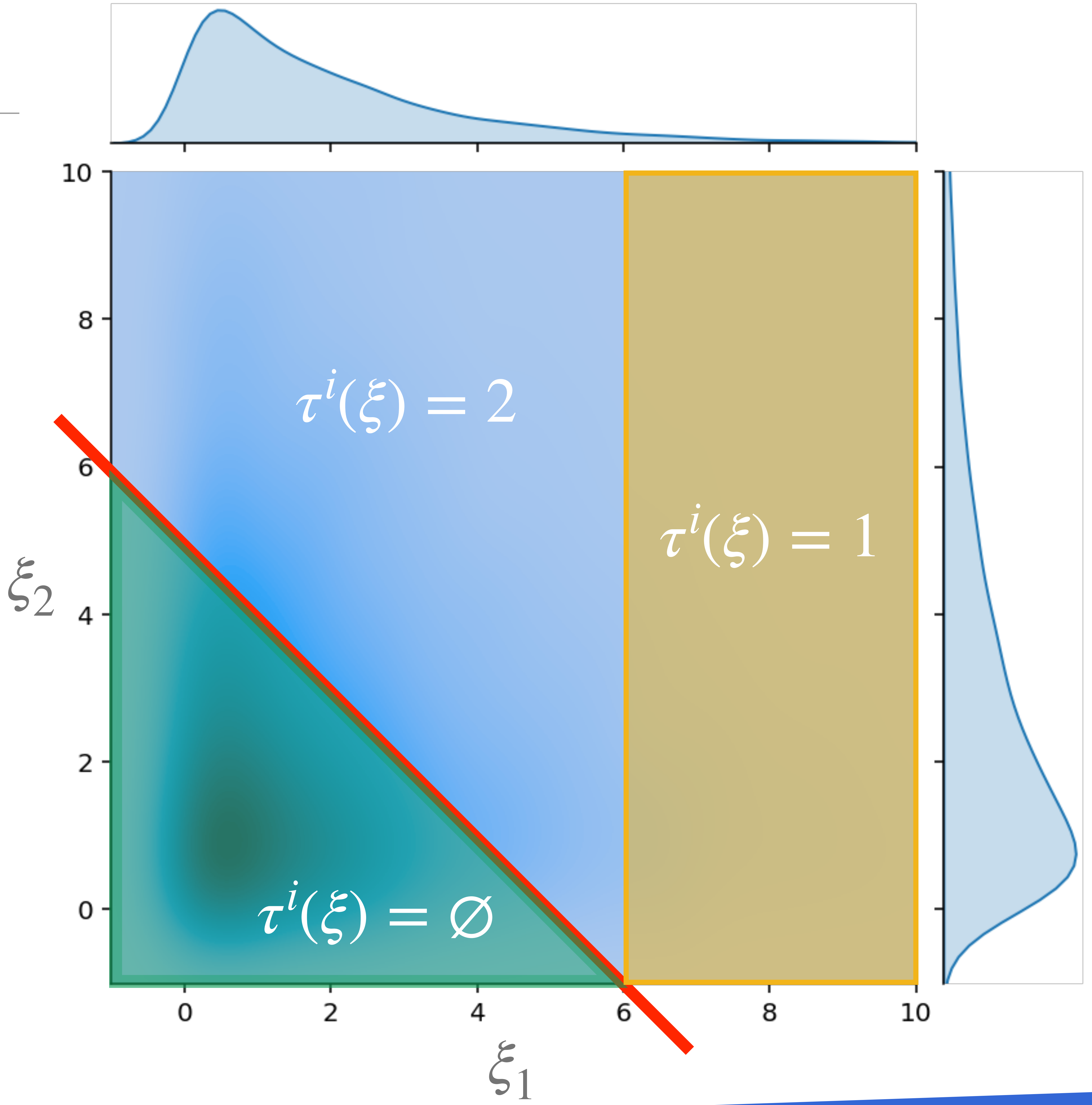


Example TO FIX

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$.

Capacity $K^i = 6$

	$S_1 \leq K$	$S_2 \leq K$	$k = 2$
$\mathbb{P}[\tau^i = k]$	0.35	0.05	0.60
$\mathbb{E}[\xi_1 \mid \tau^i = k]$	2.57	8.00	1.16
$\mathbb{E}[\xi_2 \mid \tau^i = k]$		3.00	4.16



Google Research

Benders formulation for general distributions.

Theorem: Given an incumbent solution (x^*, y^*), then

$$\Theta^i \leq \sum_{k\in J}\sum_{j\in J}(g_{ij} - c_{ij} - (g_{ik} - c_{ik}))^+ \cdot \mathbb{E}_{x^*}[\xi_j \mid \tau^i = k] \cdot \mathbb{P}_{x^*}[\tau^i = k] \cdot x_{ij}$$
$$+ \sum_{j\in J}(g_{ij} - c_{ij}) \cdot \mathbb{E}_{x^*}[\xi_j \mid \tau^i = \varnothing] \cdot \mathbb{P}_{x^*}[\tau^i = \varnothing] \cdot x_{ij}$$
$$+ \sum_{k\in J}(g_{ik} - c_{ik}) \cdot K_i \cdot y_i \cdot \mathbb{P}_{x^*}[\tau^i = k]$$

is a Bender optimality cut for the master problem.

Algorithm for general distributions

1) Solve

$$\min_{x,y} \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{j\in J} g_{ij}\mu_j x_{ij} - \sum_{i\in I}\Theta_i$$

$$x_{ij} \le y_i \qquad Ax + By \le h \qquad x, y \in \{0,1\}$$

2) Given constructed facilities and assignment of customers, for each facility sort their assigned customers by decreasing cost and construct the partition of scenarios for the different critical customers τ^i.

3) Compute $\mathbb{E}[\xi_j \,|\, \tau^i = k]$ and $\mathbb{P}[\tau^i = k]$

4) Add the violated Bender optimality cut to the main problem

$$\Theta^i \le \sum_{k\in J}\sum_{j\in J} (g_{ij} - c_{ij} - (g_{ik} - c_{ik}))^+ \cdot \mathbb{E}[\xi_j \mid \tau^i = k] \cdot \mathbb{P}[\tau^i = k] \cdot x_{ij} + \sum_{j\in J} (g_{ij} - c_{ij}) \cdot \mathbb{E}[\xi_j \mid \tau^i = \emptyset] \cdot \mathbb{P}[\tau^i = \emptyset] \cdot x_{ij} + \sum_{k\in J} (g_{ik} - c_{ik}) \cdot K_i \cdot y_i \cdot \mathbb{P}[\tau^i = k]$$

5) If no cut added, stop (optimal solution). In other case, go to (1).

Google Research

Computing $\mathbb{E}[\xi_j \,|\, \tau^i = k]$ and $\mathbb{P}[\tau^i = k]$

Bernoulli distribution with mean μ_j:

$$\mathbb{P}_{\hat{x}}[\tau^i = k] = \mu_k \cdot \mathbb{P}[S_{j<\tau_k^i} = K_i - 1]$$

$$\mathbb{E}\left[\xi_j \mid \tau^i = k\right] = \begin{cases} \mu_j & \hat{x}_{ij} = 0 \\ \mu_j & \hat{x}_{ij} = 1, j > k \\ 1 & \hat{x}_{ij} = 1, k = j \\ \frac{K_i - 1}{\eta_{\hat{x}}} & \hat{x}_{ij} = 1, j < k \\ \mu_j \cdot \frac{\mathbb{P}\left[\sum_{l:\hat{x}_{il}=1, l \neq j} \hat{x}_{il} \cdot \xi_k < K_i - 1\right]}{\mathbb{P}\left[\sum_{l:\hat{x}_{il}=1} \cdot \xi_l < K_i\right]} & \hat{x}_{ij} = 1, k = \varnothing \end{cases}$$

where $\eta_{\hat{x}}$ is the number of customers assigned to the facility.

Google Research

Computing $\mathbb{E}[\xi_j \,|\, \tau^i = k]$ and $\mathbb{P}[\tau^i = k]$

Exponential distribution with parameter $1/\mu$:

$$\mathbb{P}_{\hat{x}}[\tau^i = k] = \frac{e^{-K_i/\mu} \cdot (K_i/\mu)^{\eta_{\hat{x}}}}{\Gamma(\eta_{\hat{x}} + 1)} = \mu \cdot f_{Gamma(\eta_{\hat{x}}+1,\,\mu)}(K_i)$$

$$\mathbb{E}\left[\xi_j \mid \tau^i = k\right] = \begin{cases} \mu & \hat{x}_{ij} = 0 \\ \mu & \hat{x}_{ij} = 1, j > k \\ \mu + \frac{K_i}{\eta_{\hat{x}}+1} & \hat{x}_{ij} = 1, k = j \\ \frac{K_i}{\eta_{\hat{x}}+1} & \hat{x}_{ij} = 1, j < k \\ \mu \cdot \frac{\mathbb{P}[X_{\nu+1} \le K_i]}{\mathbb{P}[X_\nu \le K_i]} & \hat{x}_{ij} = 1, k = \varnothing \end{cases}$$

where $\eta_{\hat{x}}$ is the number of customers assigned to the facility and X_ν is a r.v. with Gamma distribution of parameters $(\eta_{\hat{x}}, \mu)$

Small instance: Bienek with i.i.d. Bernoulli demands

Using Bienek instances as Bernoulli i.i.d. demands

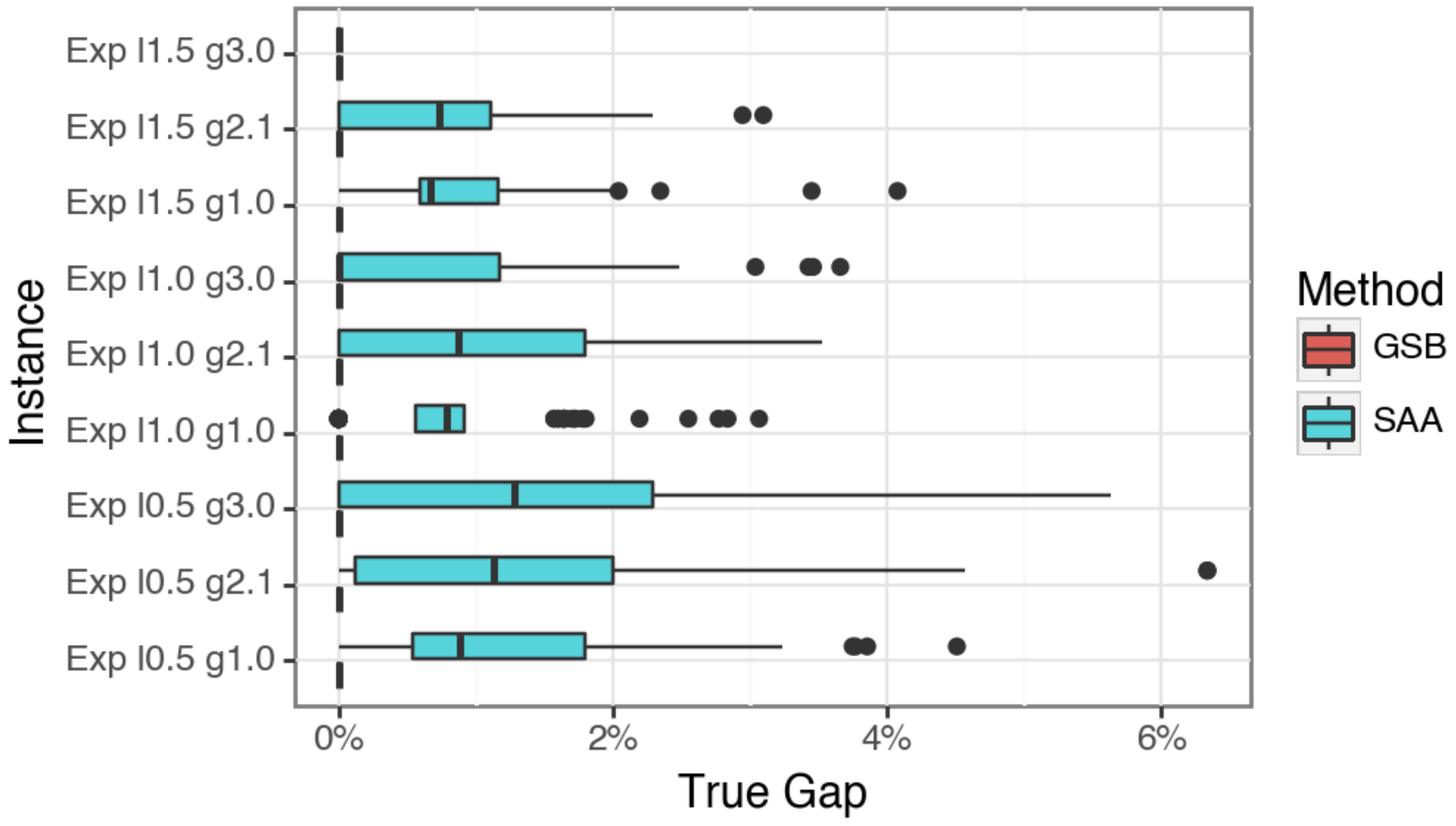
(12 customers := 4096 scenarios)

Solving time (seconds)				
Instance (λ,γ)	Det. Equiv	Bender All Scen.	Compact i.i.d.	General Bender
1.5/1.0	21.87	163.23	0.01	0.04
1.5/2.1	30.07	119.68	0.02	0.02
1.5/3.0	29.82	51.03	0.01	0.01

Even with a small instance, using all scenarios in a DE formulation is too costly.

Small instance: Bienek with i.i.d. Exponential demands

Instance (λ,γ)	Compact i.i.d.	General Bender	SAA* (Mean time)
	Solving time (seconds)		
0.5/1.0	0.02	2.23	1.47
0.5/2.1	0.04	0.15	0.27
0.5/3.0	0.01	0.09	0.09
1.0/1.0	0.02	0.55	0.33
1.0/2.1	0.01	0.09	0.15
1.0/3.0	0.02	0.03	0.14
1.5/1.0	0.01	0.12	0.12
1.5/2.1	0.02	0.13	0.09
1.5/3.0	0.03	0.00	0.12



(*) Average time over 100 repetitions with 100 scenarios

Similar times, but in most of the cases SAA does not obtain the optimal solution.

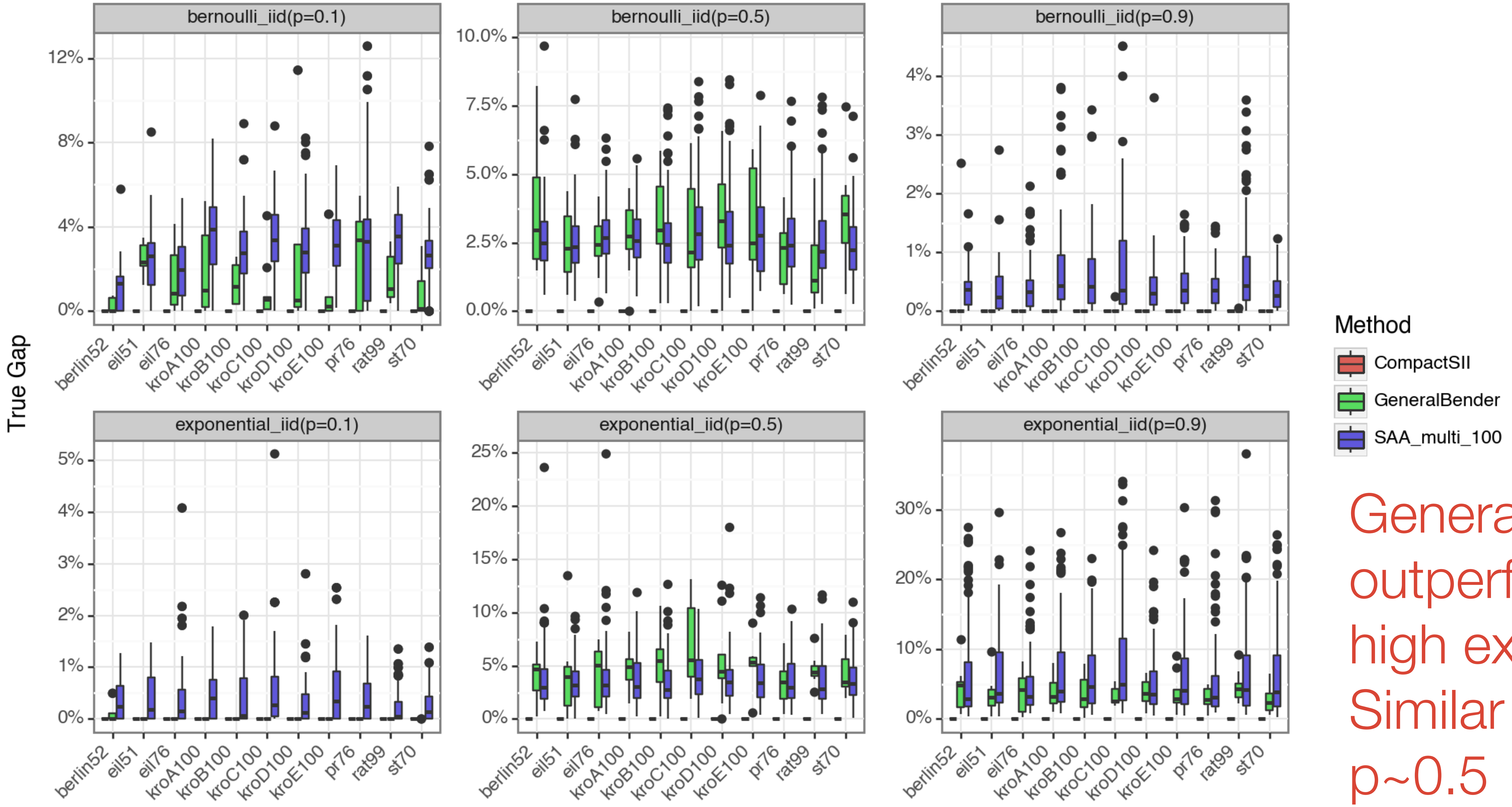
Google Research

Large Instances: Albareda-Sambola et al (2011) with i.i.d. demands

Compact formulation solves all problems in <5 minutes

Google Research

Large Instances: Albareda-Sambola et al (2011) with i.i.d. demands



General Bender outperforms SAA for low/high expected demands. Similar performance when p~0.5

Google Research

Albareda-Sambola et al (2017) with non-i.i.d. Bernoulli demands