Stochastic facility location problems with outsourcing costs

Eduardo Moreno (Google Research, France & Universidad Adolfo Ibañez, Chile)
Ivana Ljubić (ESSEC Business School of Paris, France)
Javiera Barrera (Universidad Adolfo Ibáñez, Chile)

ICSP 2025, Jul 28-Aug 1, 2025.

Google Research

Problem definition: a two-stage stochastic problem

- **First stage**: an "assignment" problem.
  - Facility location: Customers assigned to open facilities
  - Generalized Assignment: Tasks assigned to agents
  - Vehicle Routing: Customers assigned to vehicles
- **Second stage**: an stochastic demand to be served with outsourcing/penalty.
  - Customer/task demands are served by the assigned facility/agent/vehicle minimizing cost.
  - If the total demand is higher than the capacity, the unserved demands is outsourced / penalized at a higher cost.
- **Objective**: Minimize the assignment cost + expected value of serving demand.
- **This talk:** Solve the problem for general probability distributions (not scenarios)

Google Research

Facility Location

$$\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} d_j x_{ij}$$

$$x_{ij} \le y_i$$

$$(x, y) \in \mathcal{X}$$

$$x, y \in \{0,1\}$$

i

j

d_j=3

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients minimizing the
installment fix costs f_i and assignment costs g_{ij}

Facility Location

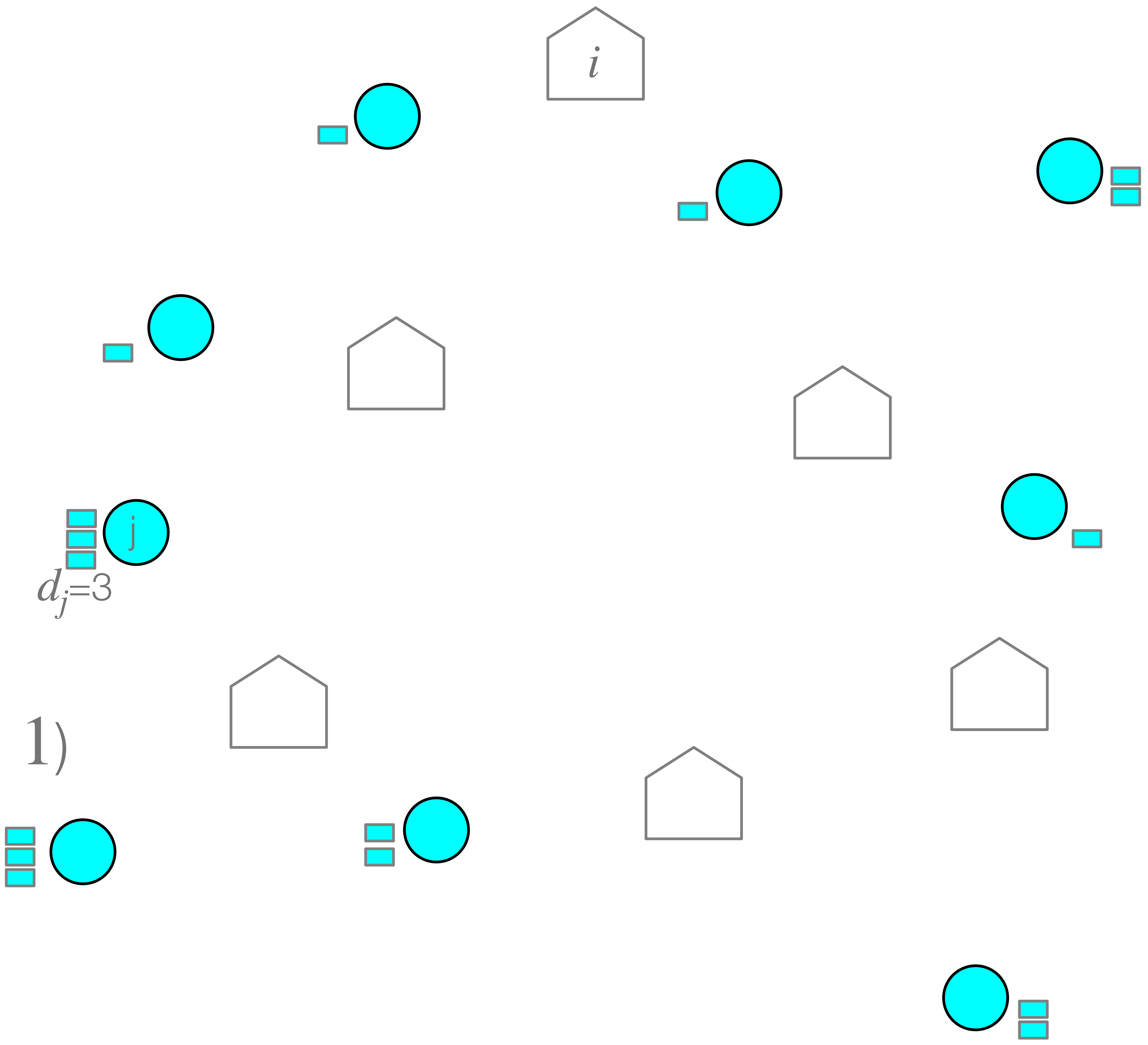
$$\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} d_j x_{ij}$$

$$x_{ij} \le y_i$$

$$(x, y) \in \mathcal{X}$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients minimizing the installment fix costs f_i and assignment costs g_{ij}



Facility Location

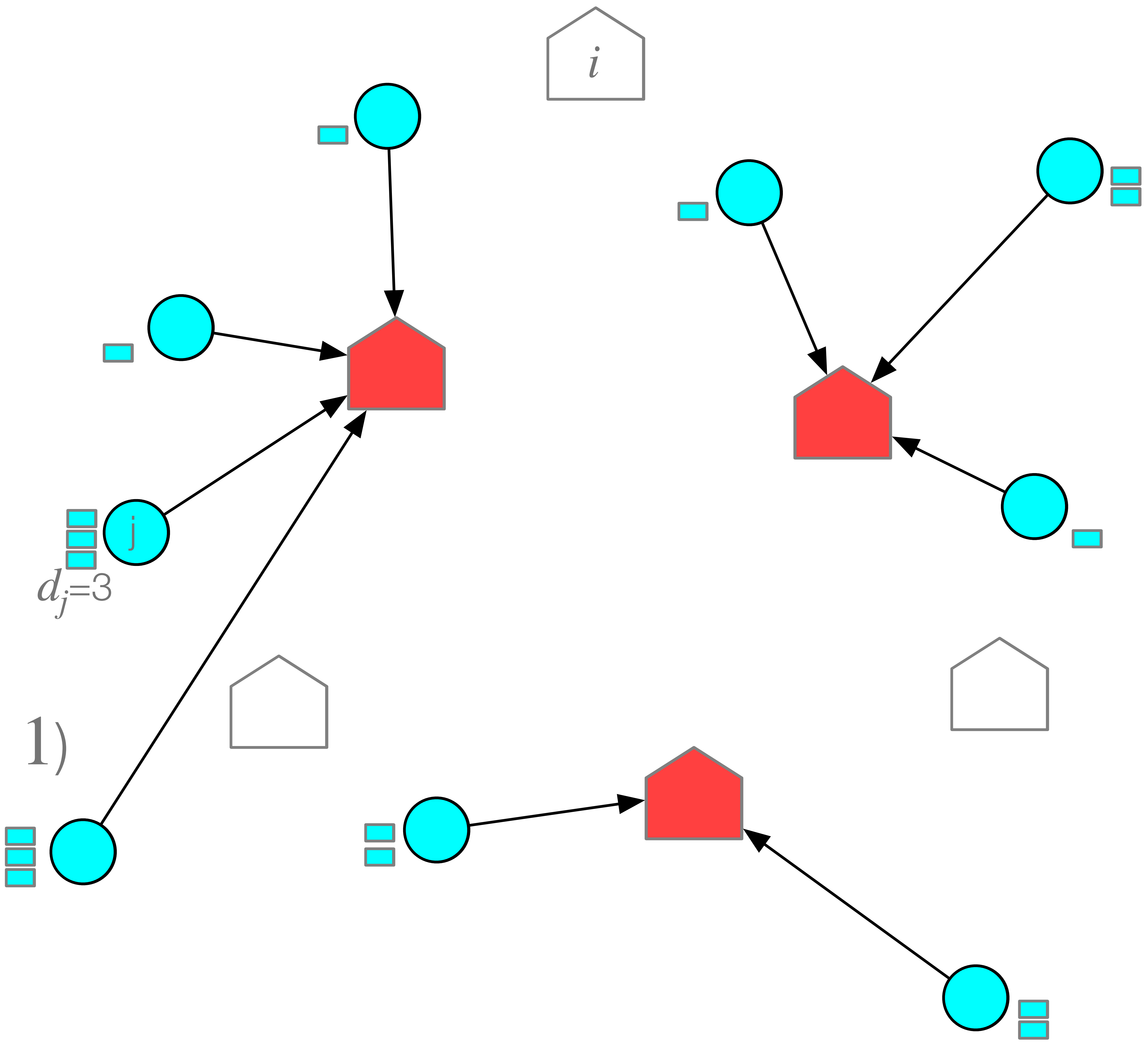
$$\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} d_j x_{ij}$$

$$x_{ij} \leq y_i$$

$$(x, y) \in \mathcal{X}$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients minimizing the
installment fix costs f_i and assignment costs g_{ij}



Facility Location

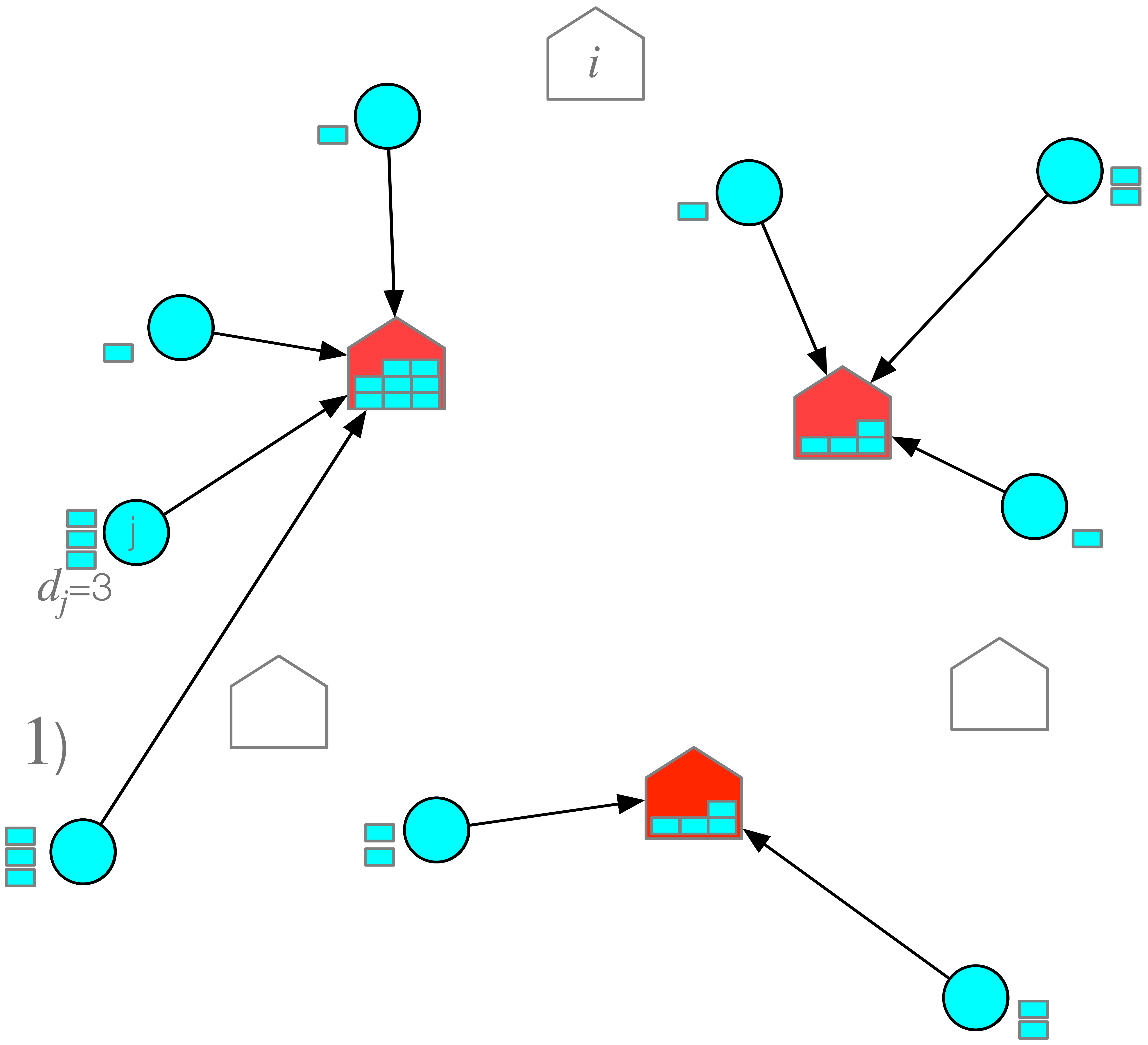
$$\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} d_j x_{ij}$$

$$x_{ij} \leq y_i$$

$$(x, y) \in \mathcal{X}$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j and a set of potential facilities I to decide a subset of facilities to open $(y_i = 1)$ and an assignment of customers to facilities $(x_{ij} = 1)$ to fulfill the demand of clients minimizing the installment fix costs f_i and assignment costs g_{ij}



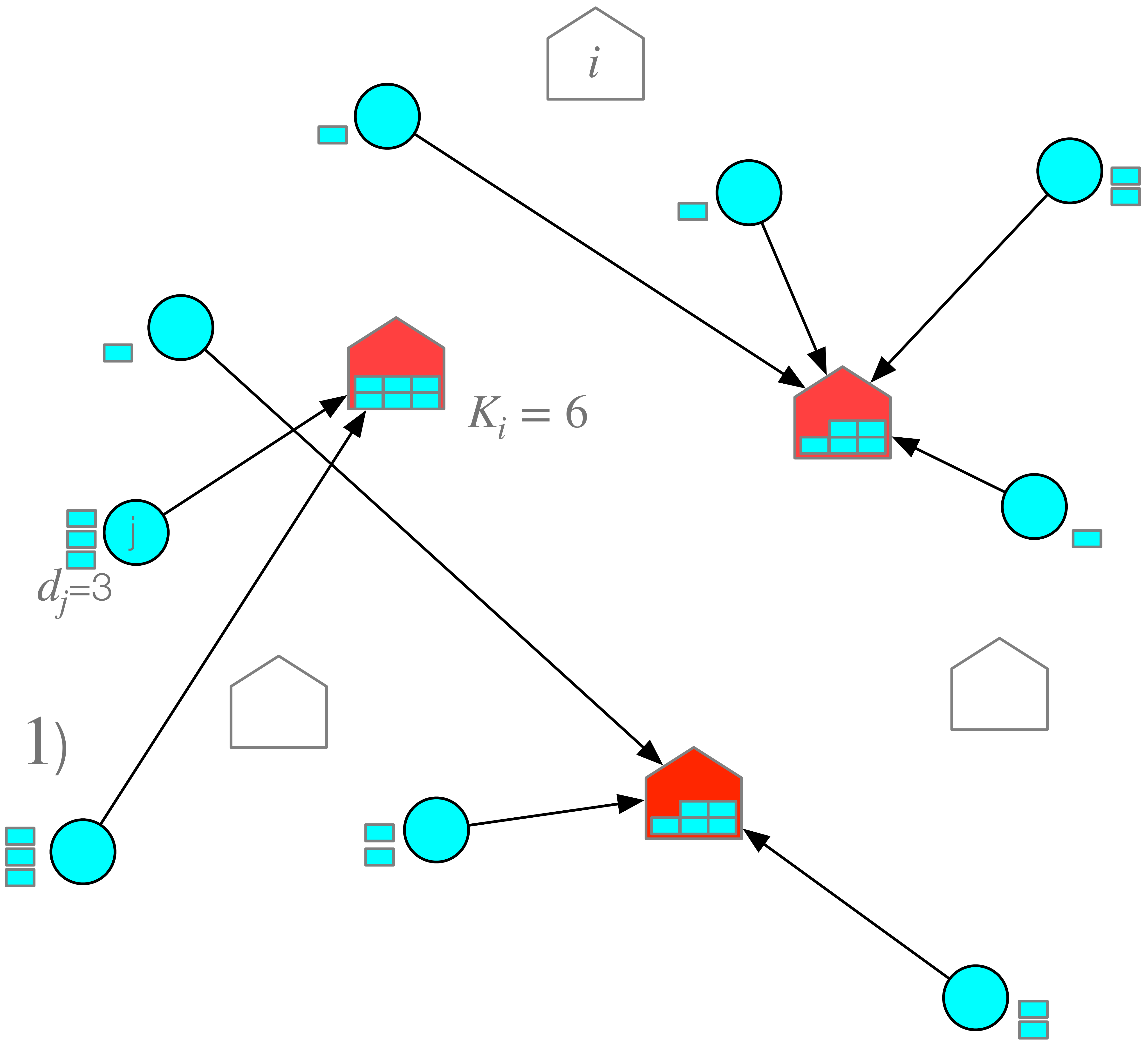
Capacitated Facility Location

$$\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} d_j x_{ij}$$

$$x_{ij} \leq y_i$$

$$(x, y) \in \mathcal{X} \qquad \sum_j d_j x_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$



Given a set of customers J with demand d_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients minimizing the
installment fix costs f_i and assignment costs g_{ij}
while satisfying the capacity of the facility K_i

Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} w_{ij} + h_{ij}\left(d_j \cdot x_{ij} - w_{ij}\right)$$

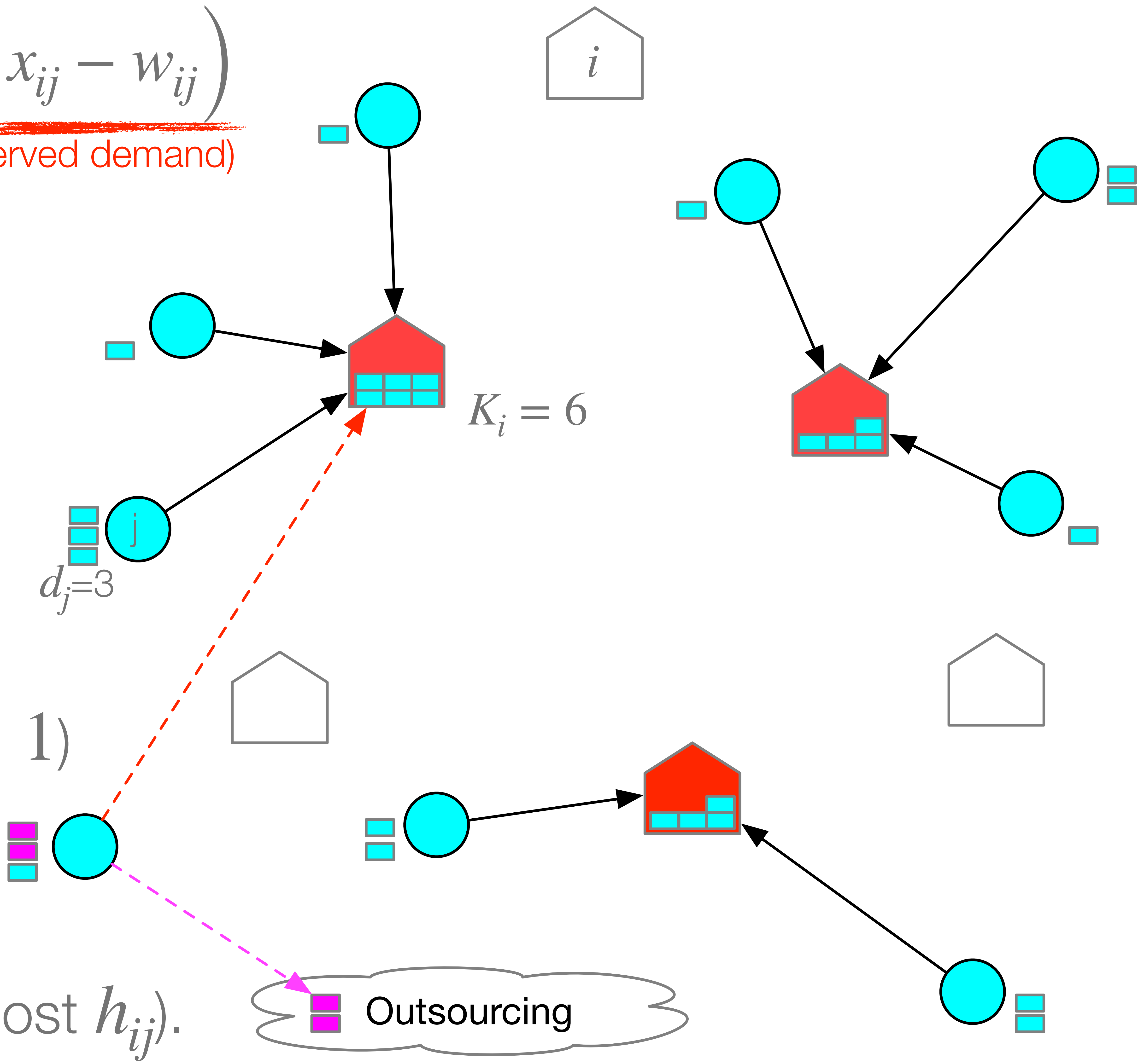
(unserved demand)

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le d_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with demand d_j and a set of potential facilities I to decide a subset of facilities to open $(y_i = 1)$ and an assignment of customers to facilities $(x_{ij} = 1)$ to fulfill the demand of clients minimizing the installment fix costs f_i and assignment costs g_{ij} while satisfying the capacity of the facility K_i allowing to outsource some demand (at a higher cost h_{ij}).



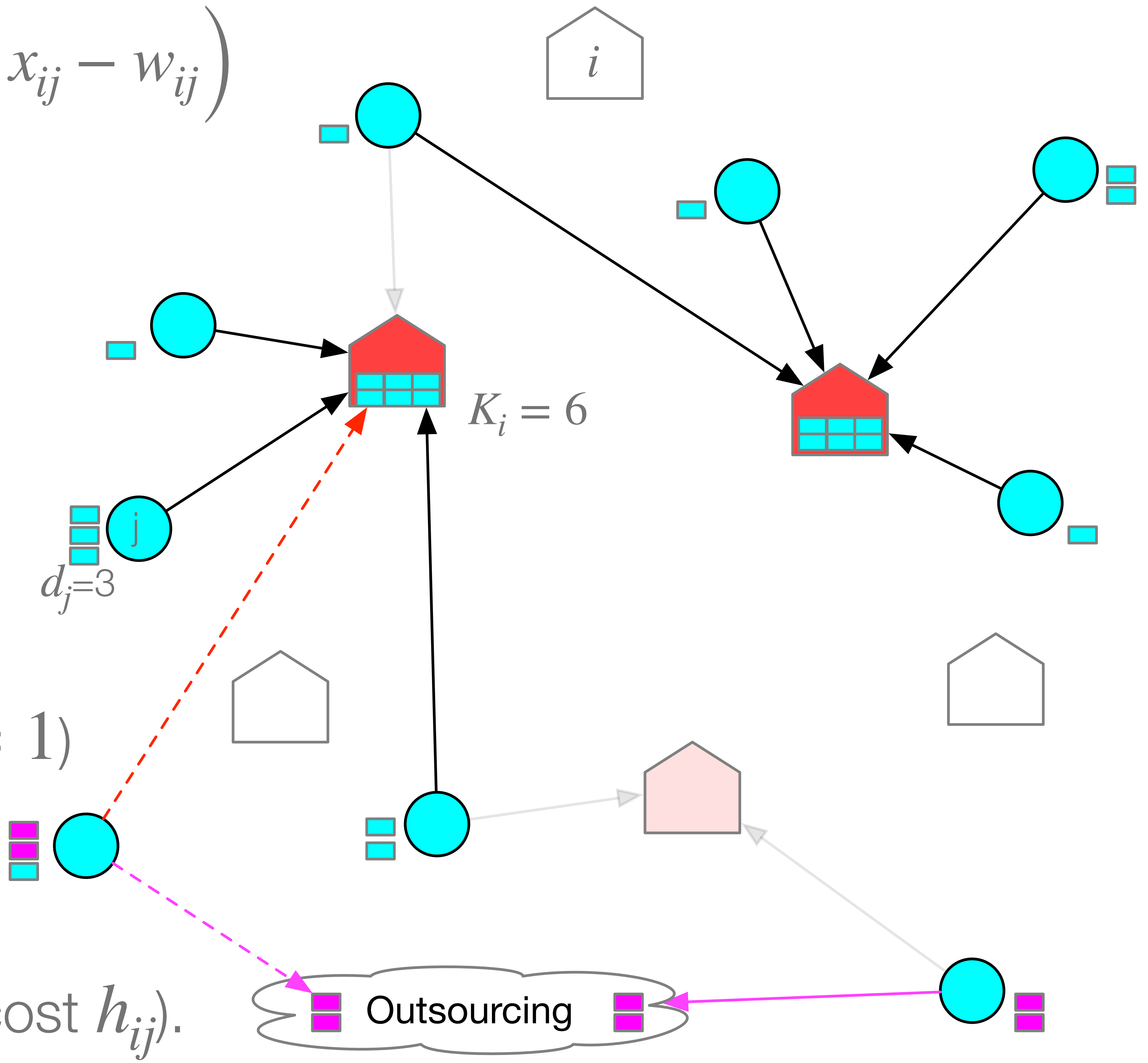
Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} w_{ij} + h_{ij}\left(d_j \cdot x_{ij} - w_{ij}\right)$$

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le d_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$



Given a set of customers J with demand d_j and a set of potential facilities I to decide a subset of facilities to open $(y_i = 1)$ and an assignment of customers to facilities $(x_{ij} = 1)$ to fulfill the demand of clients minimizing the installment fix costs f_i and assignment costs g_{ij} while satisfying the capacity of the facility K_i allowing to outsource some demand (at a higher cost h_{ij}).

Google Research

Stochastic Capacitated Facility Location with Outsourcing

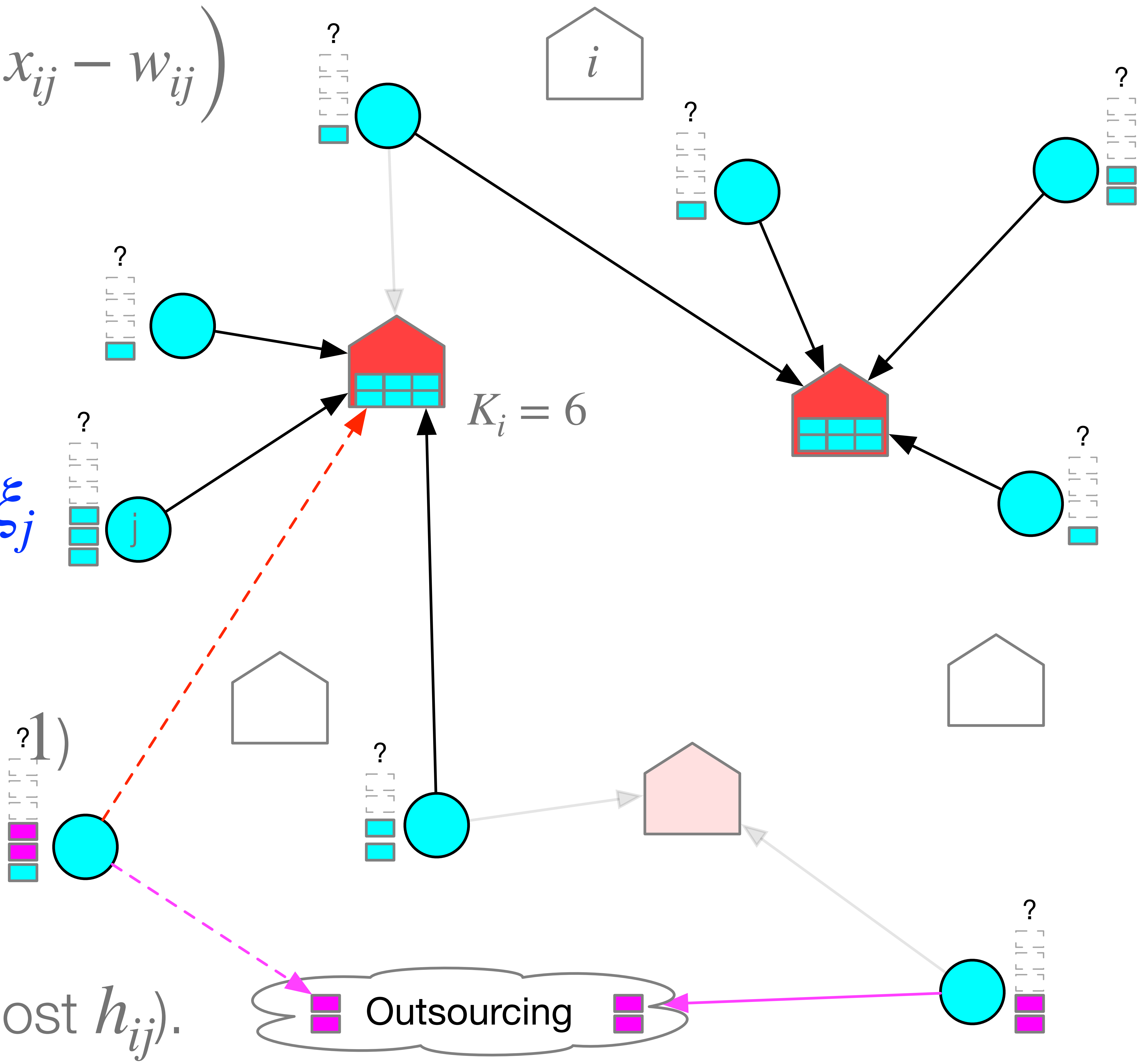
$$\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} w_{ij} + h_{ij} \left(\xi_j \cdot x_{ij} - w_{ij} \right)$$

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with random demand ξ_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients minimizing the
installment fix costs f_i and assignment costs g_{ij}
while satisfying the capacity of the facility K_i
allowing to outsource some demand (at a higher cost h_{ij}).



Stochastic Capacitated Facility Location with Outsourcing

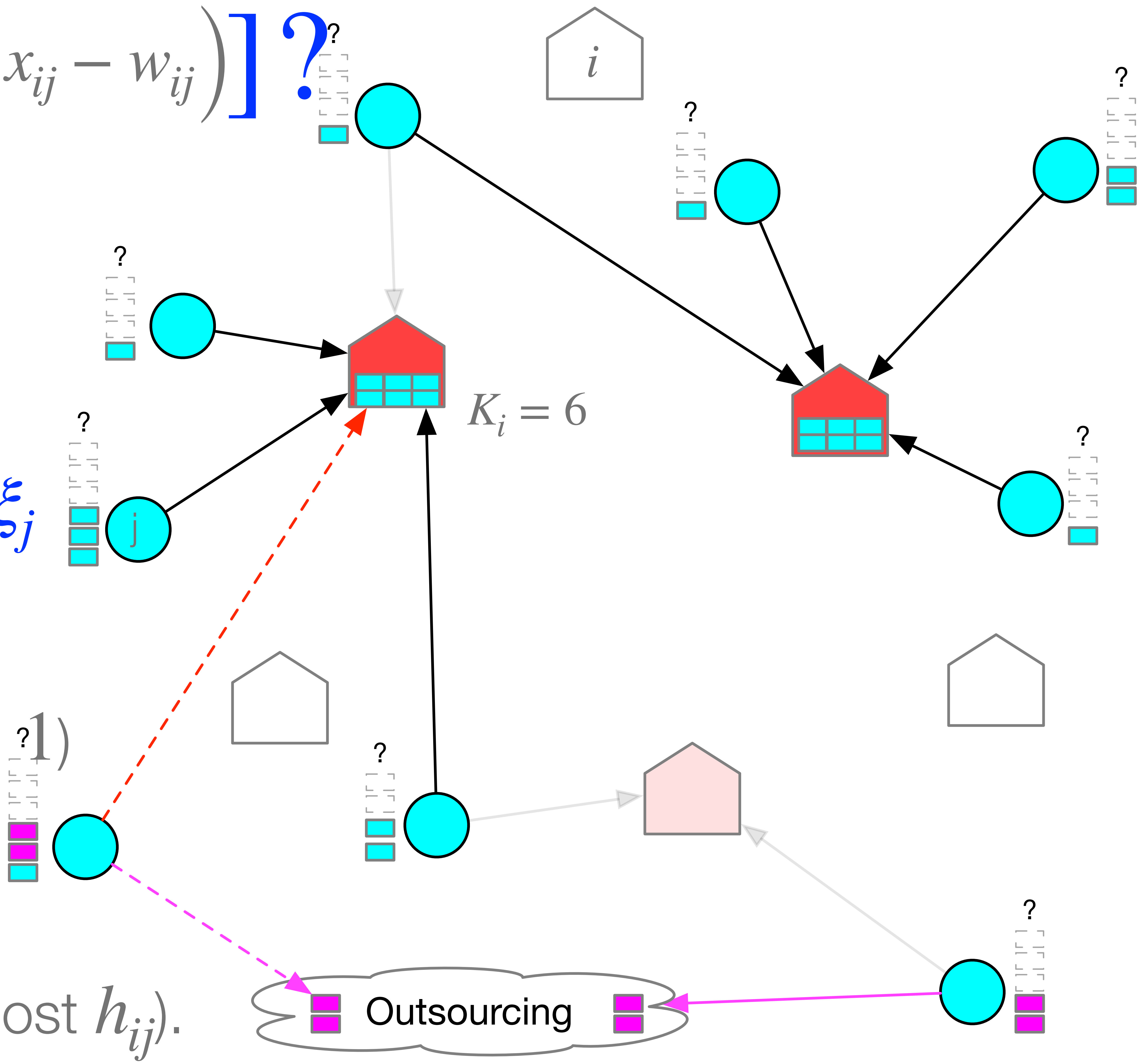
$$\mathbb{E}\left[\min \sum_i f_i y_i + \sum_i \sum_j g_{ij} w_{ij} + h_{ij} \left(\xi_j \cdot x_{ij} - w_{ij} \right) \right] ?$$

$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

Given a set of customers J with random demand ξ_j
and a set of potential facilities I
to decide a subset of facilities to open $(y_i = 1)$
and an assignment of customers to facilities $(x_{ij} = 1)$
to fulfill the demand of clients minimizing the installment fix costs f_i and assignment costs g_{ij}
while satisfying the capacity of the facility K_i
allowing to outsource some demand (at a higher cost h_{ij}).



Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mathbb{E}\left[g_{ij} w_{ij} + h_{ij} \left(\xi_j \cdot x_{ij} - w_{ij} \right) \right]$$

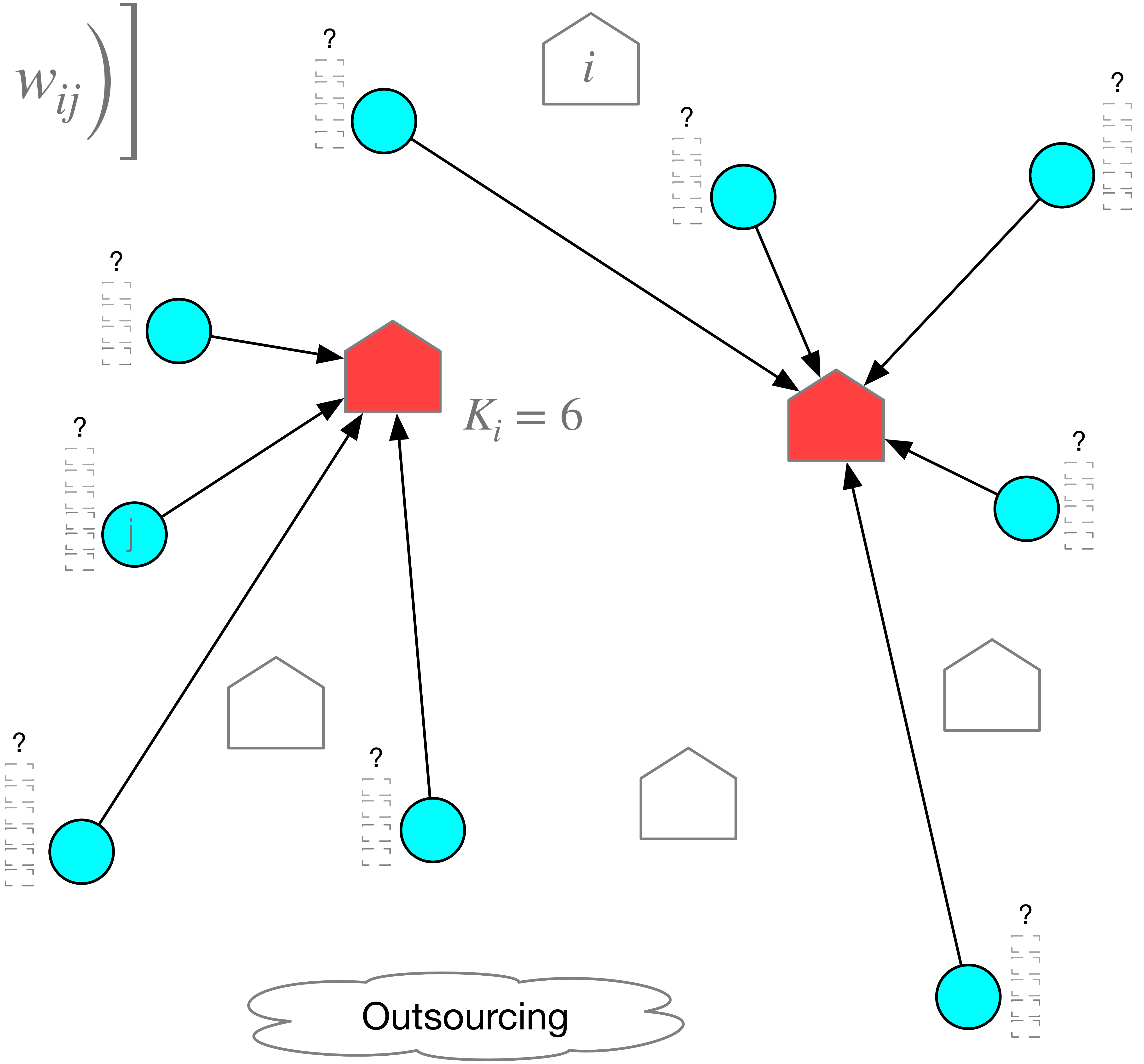
$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.



Google Research

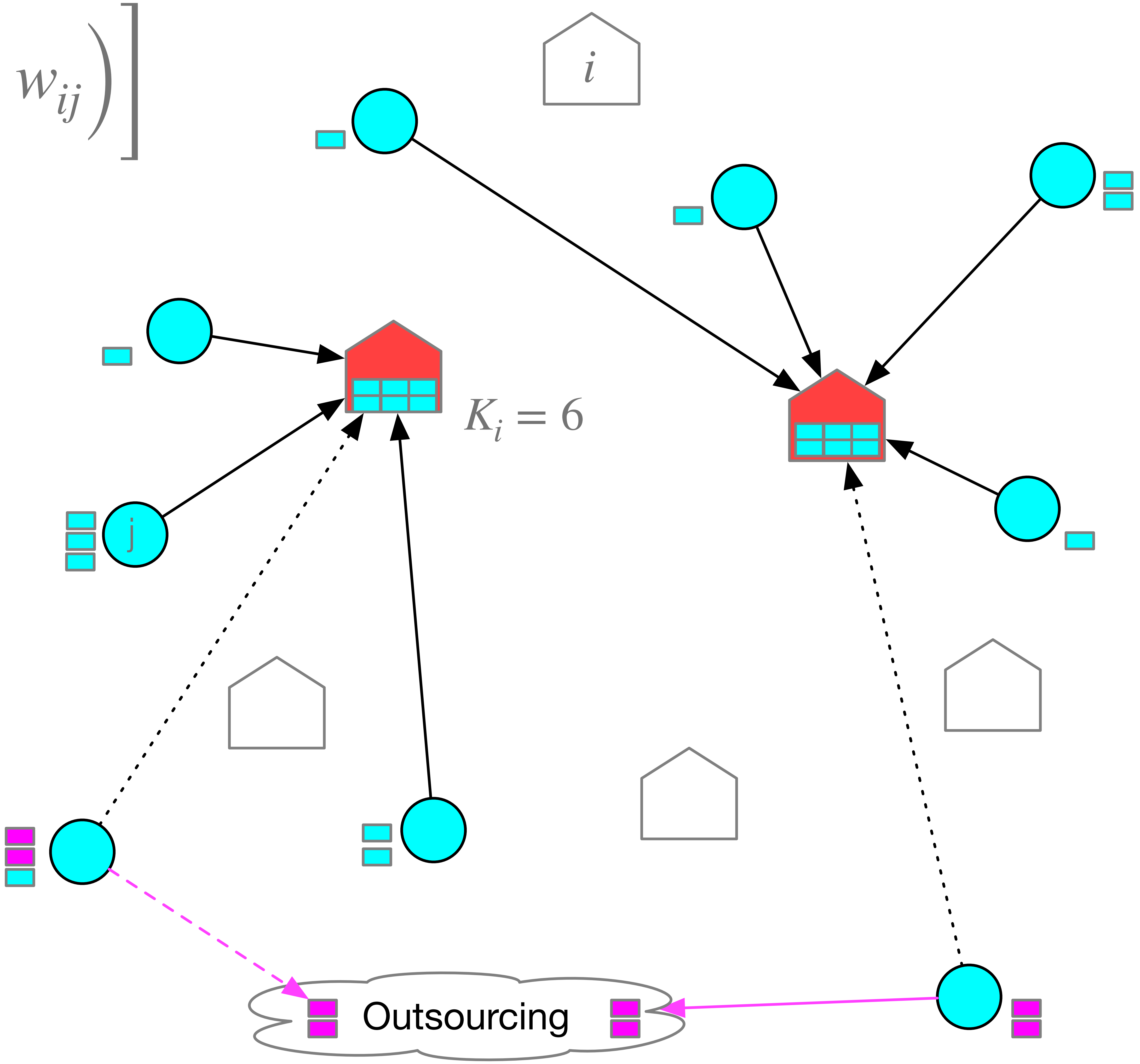
Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mathbb{E}\left[g_{ij} w_{ij} + h_{ij}\left(\xi_j \cdot x_{ij} - w_{ij}\right)\right]$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathscr{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$



We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.
- **2nd stage decision (wait-and-see):** to route and/or outsource the random demand.

Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mathbb{E}\left[g_{ij}w_{ij} + h_{ij}\left(\xi_j \cdot x_{ij} - w_{ij}\right)\right]$$

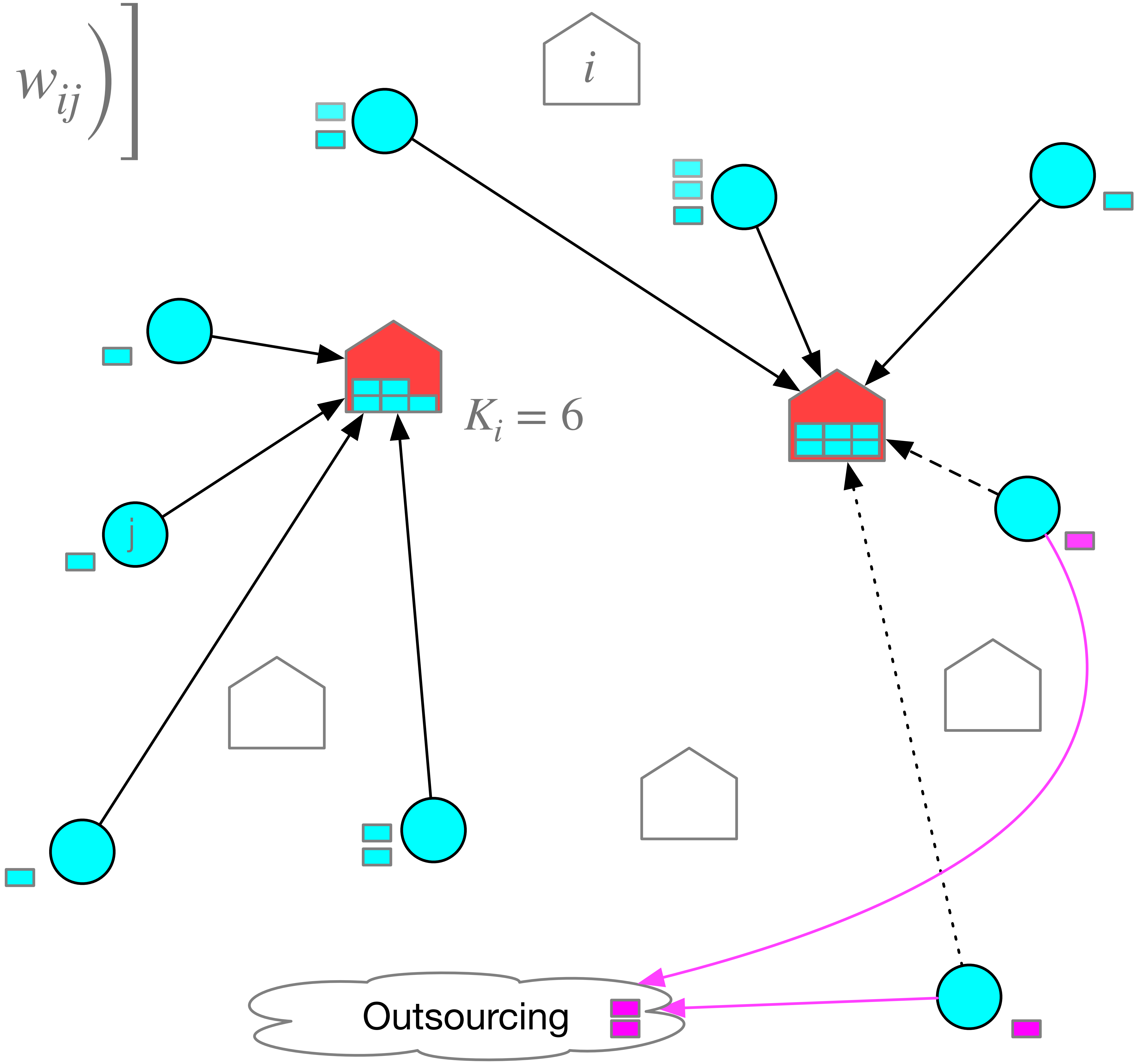
$$x_{ij} \le y_i \qquad 0 \le w_{ij} \le \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathcal{X} \qquad \sum_j w_{ij} \le K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.
- **2nd stage decision (wait-and-see):** to route and/or outsource the random demand.



Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mu_j h_{ij} x_{ij} - \sum_i \mathbb{E}\left[\sum_j \underbrace{(h_{ij} - g_{ij})}_{c_{ij}} w_{ij} \right]$$

$$x_{ij} \leq y_i \qquad 0 \leq w_{ij} \leq \xi_j \cdot x_{ij}$$

$$(x, y) \in \mathscr{X} \qquad \sum_j w_{ij} \leq K_i \cdot y_i$$

$$x, y \in \{0,1\}$$

We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.
- **2nd stage decision (wait-and-see):** to route and/or outsource the random demand.

Stochastic Capacitated Facility Location with Outsourcing

$$\min \sum_i f_i y_i + \sum_i \sum_j \mu_j h_{ij} x_{ij} - \sum_i \mathbb{E}[Q^i(x, y, \xi)]$$

$$x_{ij} \leq y_i$$
$$(x, y) \in \mathcal{X}$$
$$x, y \in \{0,1\}$$

First-stage problem

$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j c_{ij} w_{ij}$$
$$w_{ij} \leq \xi_j \cdot x_{ij}$$
$$\sum_j w_{ij} \leq K_i \cdot y_i$$

Second-stage problem (independent for each facility)

We assume a two-stage stochastic problem:

- **1st stage decision (here-and-now):** to open facilities and assign customers to them.
- **2nd stage decision (wait-and-see):** to route and/or outsource the random demand.

Application example.

Partnership between Google and Ignite Energy Access to deploy solar panels in Africa, with focus on remote rural areas.

Bloomberg

A mini-grid provides power to the community in Nyimba Mwana village in the Eastern Province of Zambia, on Feb. 20.

THE MISSION TO ELECTRIFY AFRICA MIGHT FINALLY BE UNDER WAY

Plunging solar panel prices and international funding are now spurring the rollout of so-called mini grids that can transform rural communities

Forbes

MONEY > MARKETS

How Startups Are Bringing Affordable, Clean Energy To Africa

By Gaurav Sharma, Senior Contributor. ⓘ Gaurav Sharma is a London-based a... Follow Author

Apr 22, 2025, 05:24am EDT

Share Save Comment 0

An Ignite Energy Access solar power installation in Rwanda.
IGNITE ENERGY ACCESS

(Images from https://igniteaccess.com/media/)

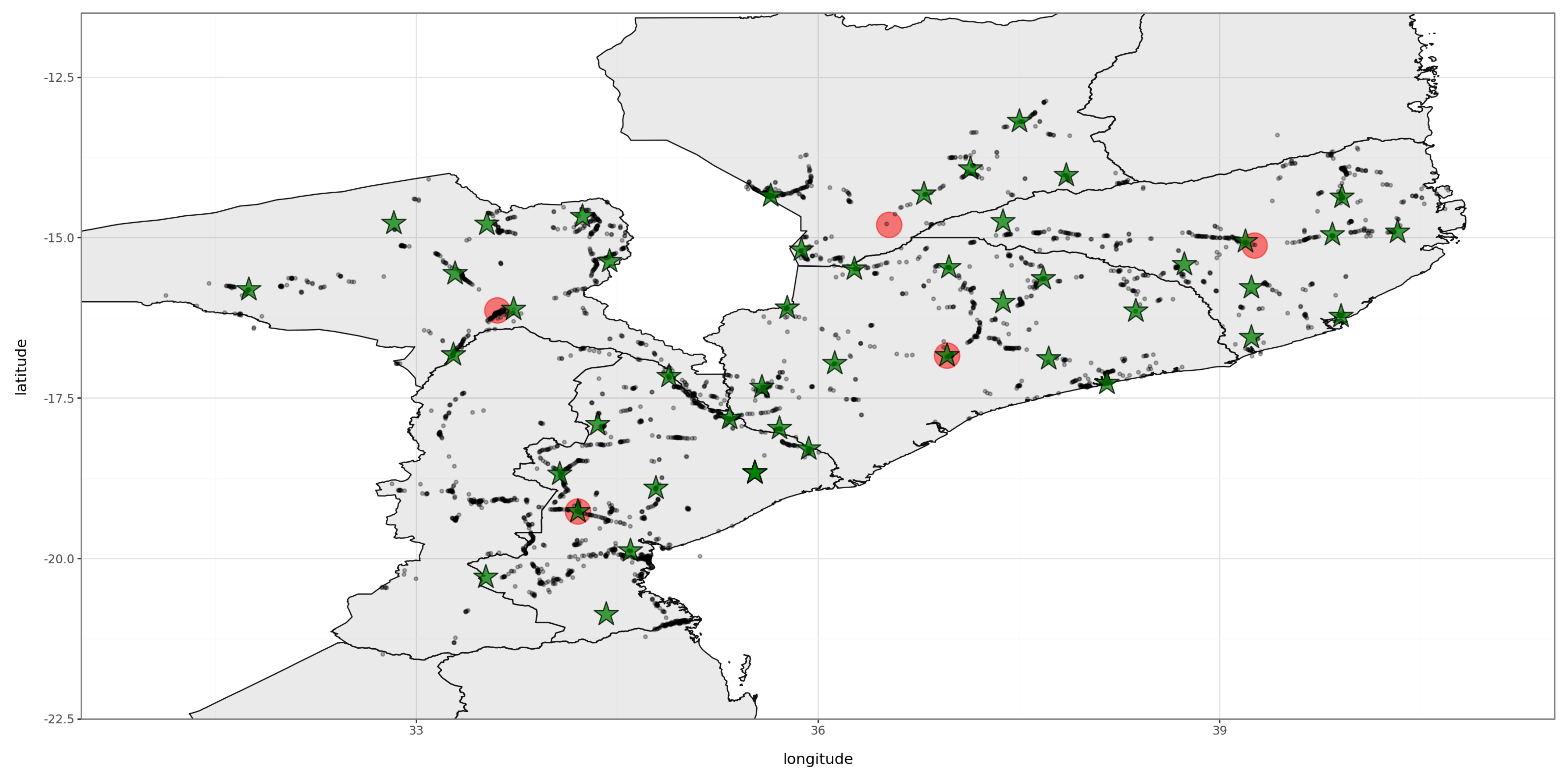
Application example.

- Facilities: “minihubs” with containers of solar panels. Very expensive due to the lack of roads.

- Customers: small villages in the country side of Mozambique.

- Agents visit customers by foot or bikes due the lack of roads.

- Uncertainty: adoption of the technology in the villages.



Stochastic facility location problems with outsourcing costs

1. **Bender formulation for a discrete set of scenarios (for example, a sample average approximation of the demand distributions)**
2. Bender formulation for general distributions.
3. Strengthened formulation.
4. Computational experiments

Second-stage problem : Knapsack problem

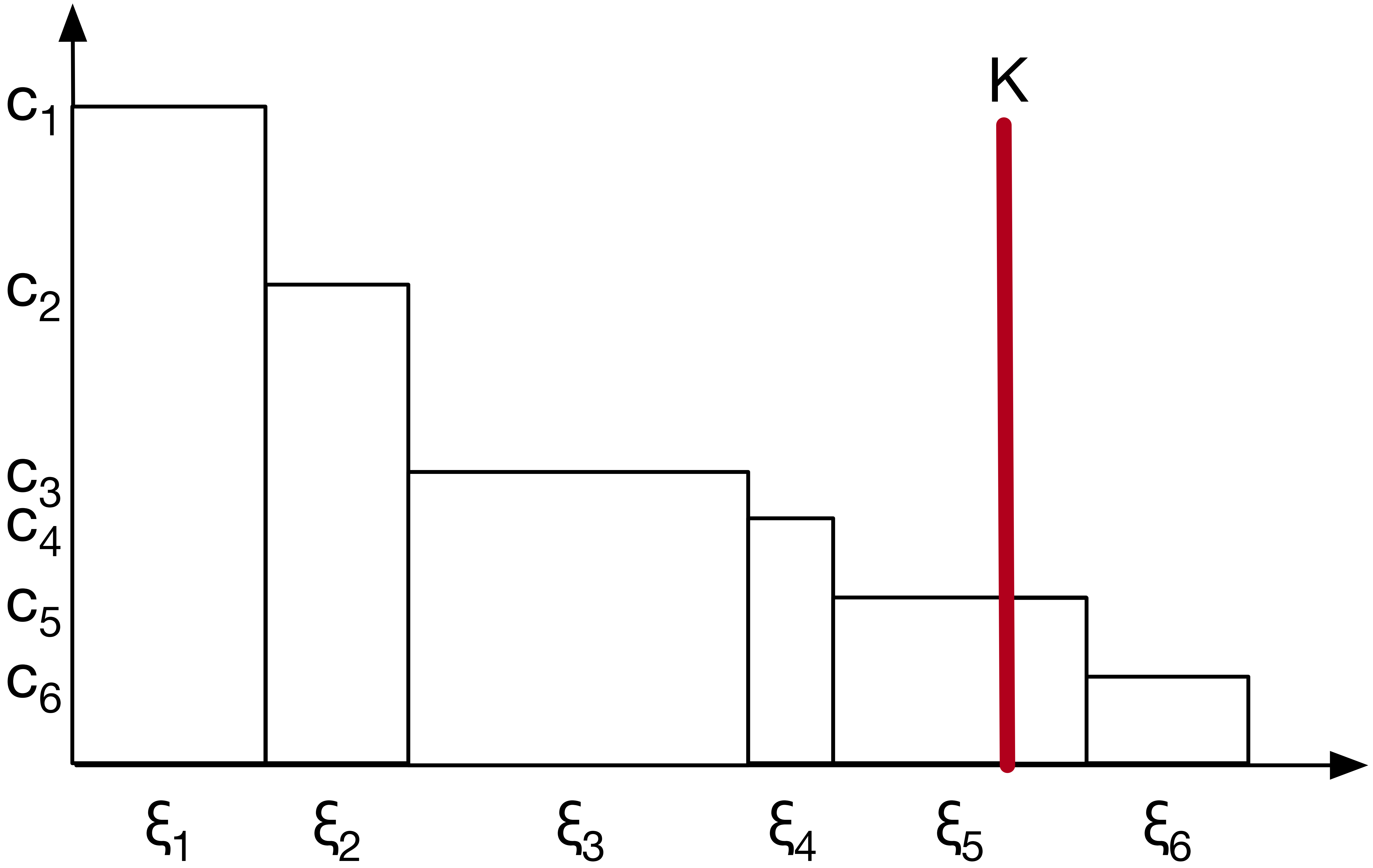
$$Q^i(x,y,\xi) = \max_{w \geq 0} \sum_j c_{ij} w_{ij}$$

$$w_{ij} \leq \xi_j \cdot x_{ij}$$

$$\sum_j w_{ij} \leq K_i \cdot y_i$$

Optimal solution: to allocate the demand in decreasing order of profit until the capacity of the facility is fulfilled.

$$w_{ij} = \begin{cases} \xi_j x_{ij} & j < \tau^i \\ K_i y_i - \sum_{l<\tau^i} x_{il}\xi_l & j = \tau^i \\ 0 & j > \tau^i \end{cases}$$



Notation assumption: indices j are already in decreasing order of profit.

Second-stage problem : Knapsack problem

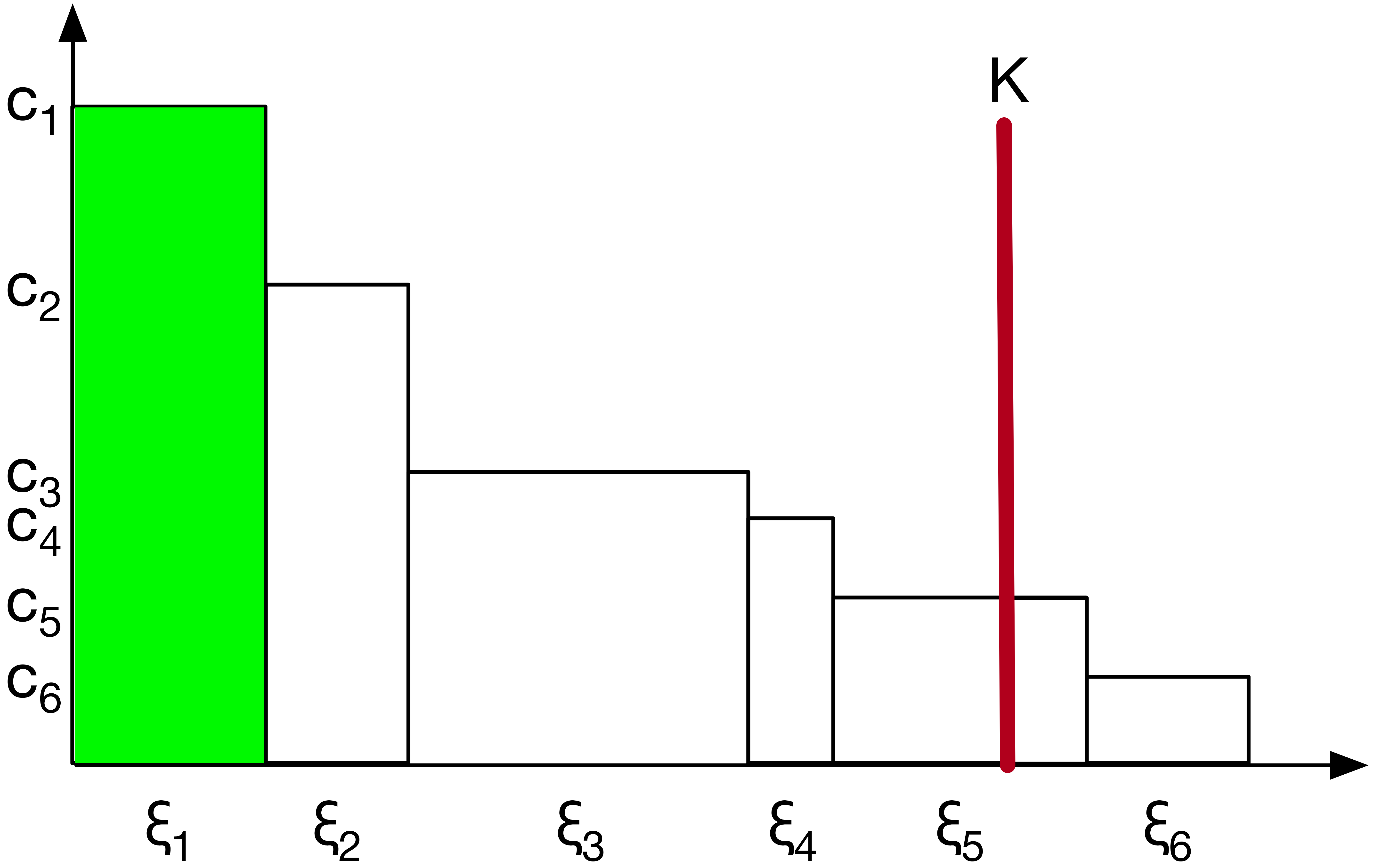
$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j c_{ij} w_{ij}$$

$$w_{ij} \leq \xi_j \cdot x_{ij}$$

$$\sum_j w_{ij} \leq K_i \cdot y_i$$

Optimal solution: to allocate the demand in decreasing order of profit until the capacity of the facility is fulfilled.

$$w_{ij} = \begin{cases} \xi_j x_{ij} & j < \tau^i \\ K_i y_i - \sum_{l<\tau^i} x_{il}\xi_l & j = \tau^i \\ 0 & j > \tau^i \end{cases}$$



Notation assumption: indices j are already in decreasing order of profit.

Second-stage problem : Knapsack problem

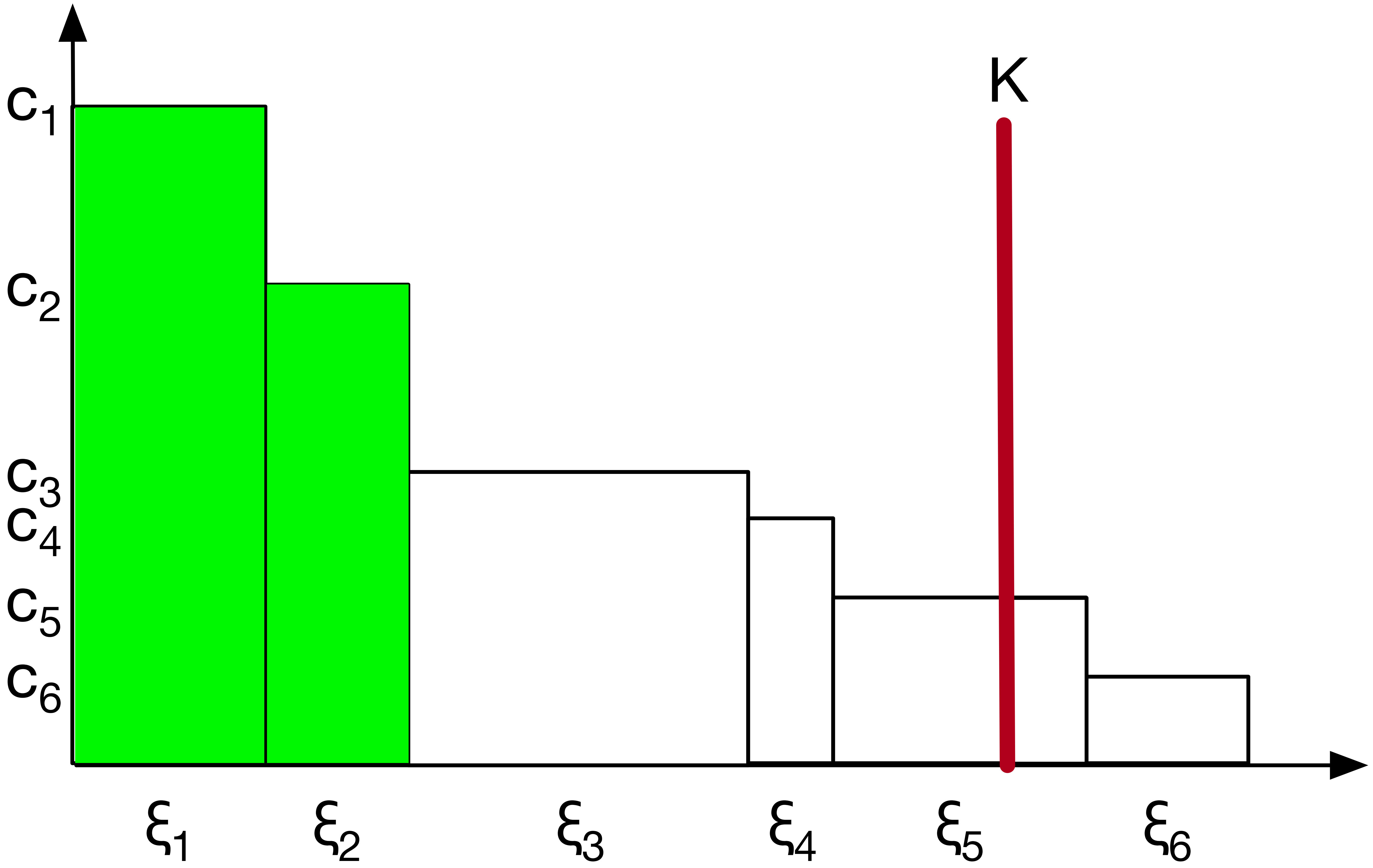
$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j c_{ij} w_{ij}$$

$$w_{ij} \leq \xi_j \cdot x_{ij}$$

$$\sum_j w_{ij} \leq K_i \cdot y_i$$

Optimal solution: to allocate the demand in decreasing order of profit until the capacity of the facility is fulfilled.

$$w_{ij} = \begin{cases} \xi_j x_{ij} & j < \tau^i \\ K_i y_i - \sum_{l<\tau^i} x_{il}\xi_l & j = \tau^i \\ 0 & j > \tau^i \end{cases}$$



Notation assumption: indices j are already in decreasing order of profit.

Second-stage problem : Knapsack problem

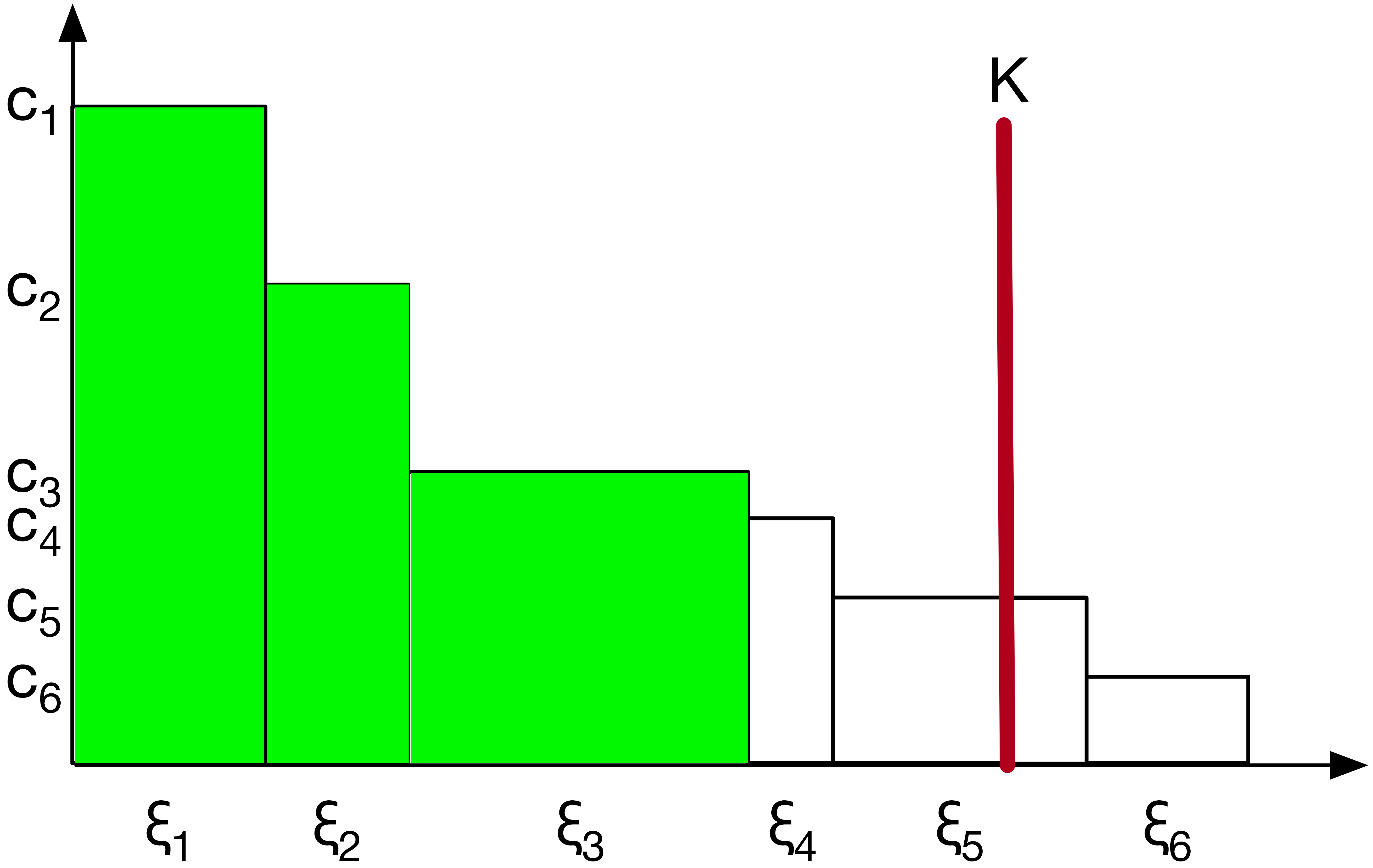
$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j c_{ij} w_{ij}$$

$$w_{ij} \leq \xi_j \cdot x_{ij}$$

$$\sum_j w_{ij} \leq K_i \cdot y_i$$

Optimal solution: to allocate the demand in decreasing order of profit until the capacity of the facility is fulfilled.

$$w_{ij} = \begin{cases} \xi_j x_{ij} & j < \tau^i \\ K_i y_i - \sum_{l < \tau^i} x_{il} \xi_l & j = \tau^i \\ 0 & j > \tau^i \end{cases}$$



Notation assumption: indices j are already in decreasing order of profit.

Google Research

Second-stage problem : Knapsack problem

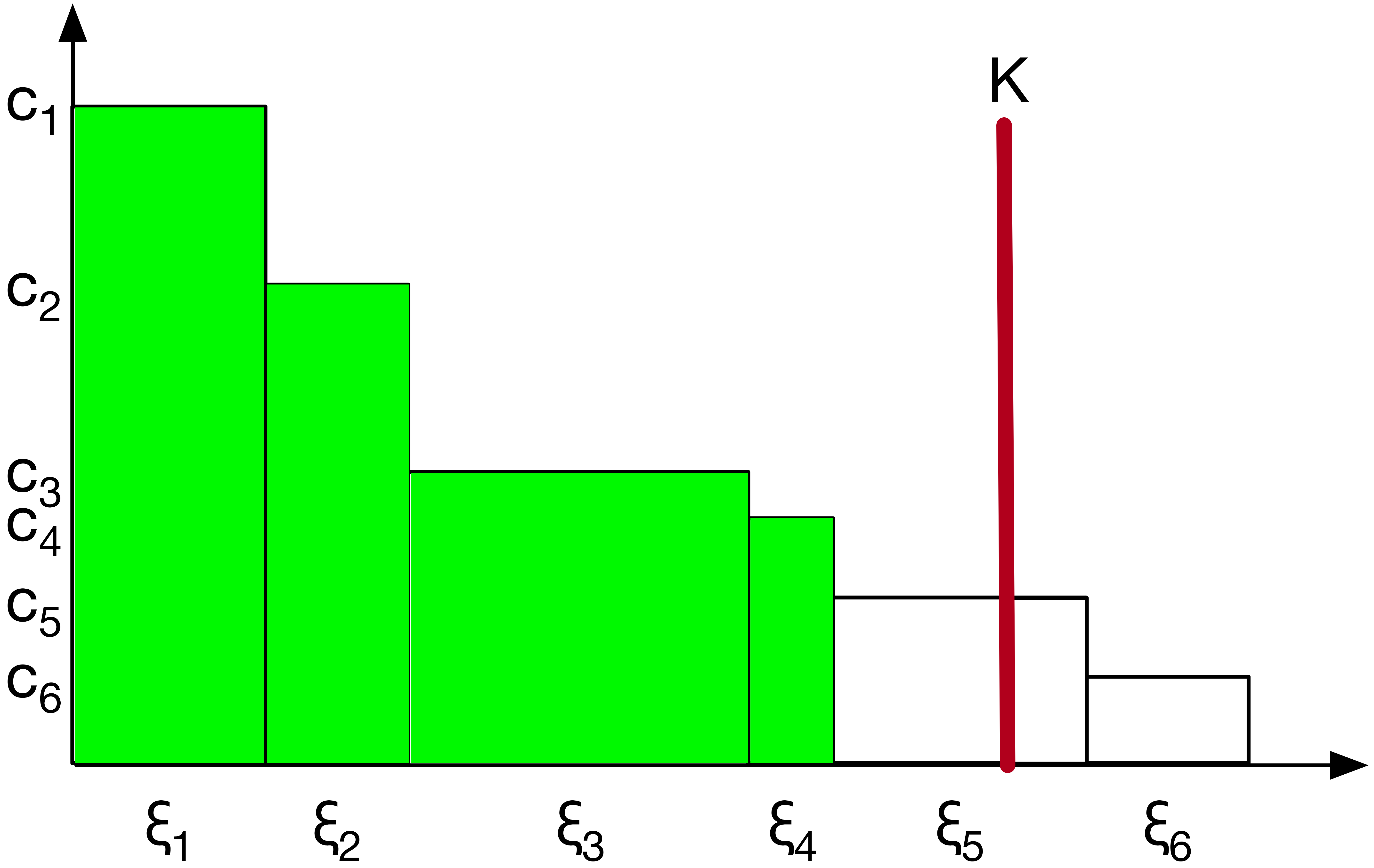
$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j c_{ij} w_{ij}$$

$$w_{ij} \leq \xi_j \cdot x_{ij}$$

$$\sum_j w_{ij} \leq K_i \cdot y_i$$

Optimal solution: to allocate the demand in decreasing order of profit until the capacity of the facility is fulfilled.

$$w_{ij} = \begin{cases} \xi_j x_{ij} & j < \tau^i \\ K_i y_i - \sum_{l<\tau^i} x_{il} \xi_l & j = \tau^i \\ 0 & j > \tau^i \end{cases}$$



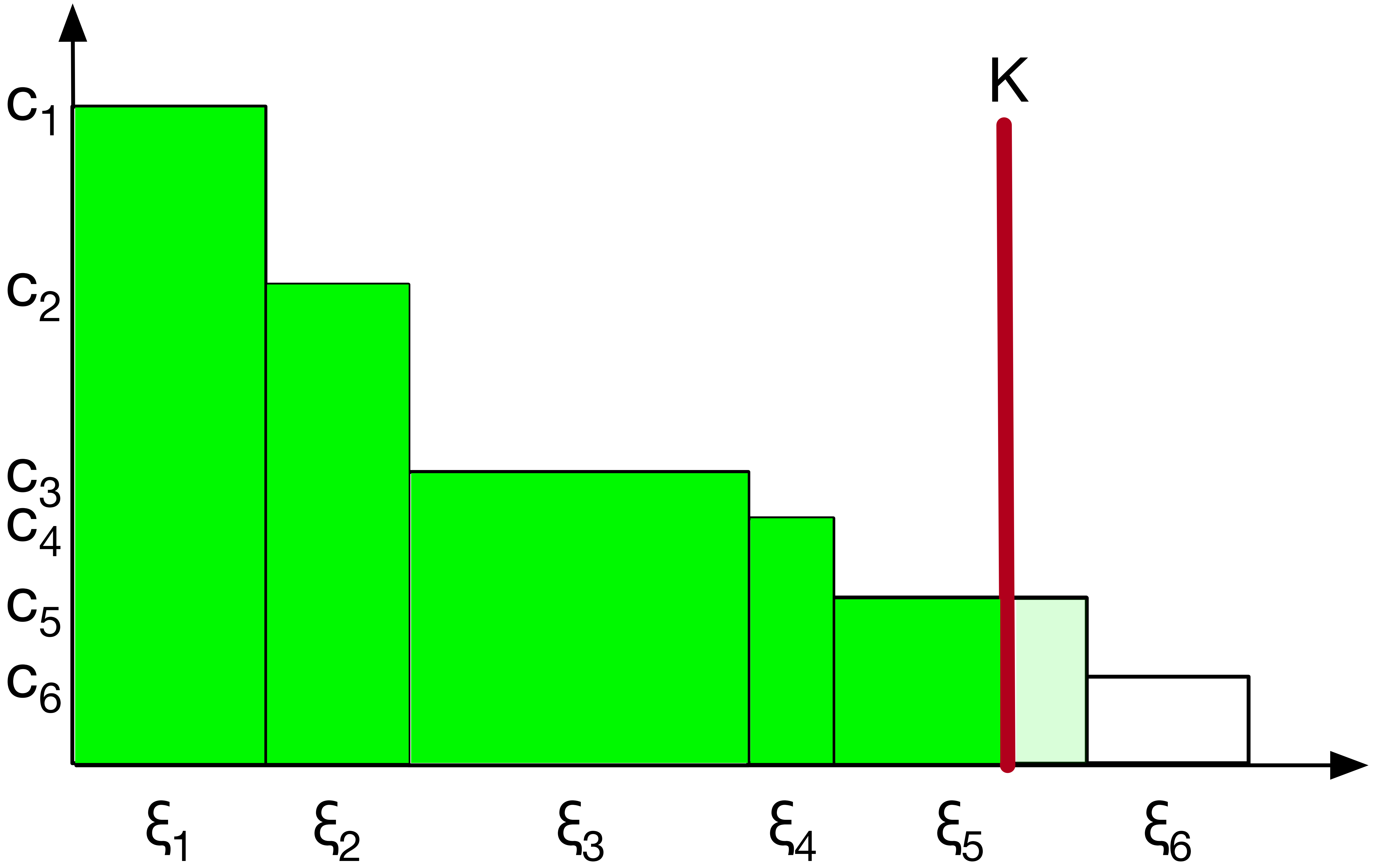
Notation assumption: indices j are already in decreasing order of profit.

Second-stage problem : Knapsack problem

$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j c_{ij} w_{ij}$$
$$w_{ij} \leq \xi_j \cdot x_{ij}$$
$$\sum_j w_{ij} \leq K_i \cdot y_i$$

Optimal solution: to allocate the demand in decreasing order of profit until the capacity of the facility is fulfilled.

$$w_{ij} = \begin{cases} \xi_j x_{ij} & j < \tau^i \\ K_i y_i - \sum_{l<\tau^i} x_{il}\xi_l & j = \tau^i \\ 0 & j > \tau^i \end{cases}$$



Notation assumption: indices j are already in decreasing order of profit.

Second-stage problem : Knapsack problem

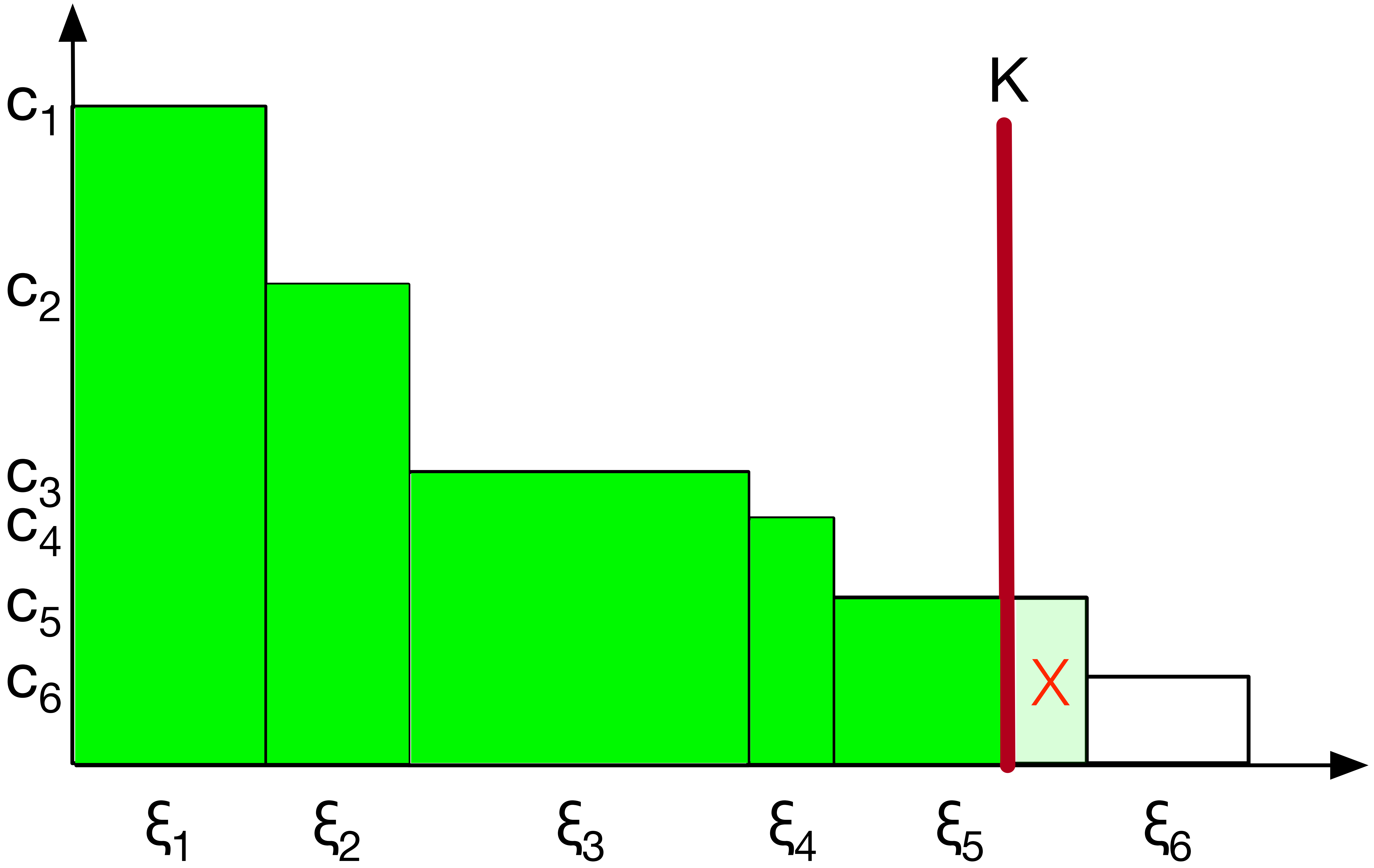
$$Q^i(x, y, \xi) = \max_{w \geq 0} \sum_j c_{ij} w_{ij}$$

$$w_{ij} \leq \xi_j \cdot x_{ij}$$

$$\sum_j w_{ij} \leq K_i \cdot y_i$$

Optimal solution: to allocate the demand in decreasing order of profit until the capacity of the facility is fulfilled.

$$w_{ij} = \begin{cases} \xi_j x_{ij} & j < \tau^i \\ K_i y_i - \sum_{l<\tau^i} x_{il}\xi_l & j = \tau^i \\ 0 & j > \tau^i \end{cases}$$



Notation assumption: indices j are already in decreasing order of profit.

Google Research

Dual solution of the subproblem

Dual of the subproblem is given by

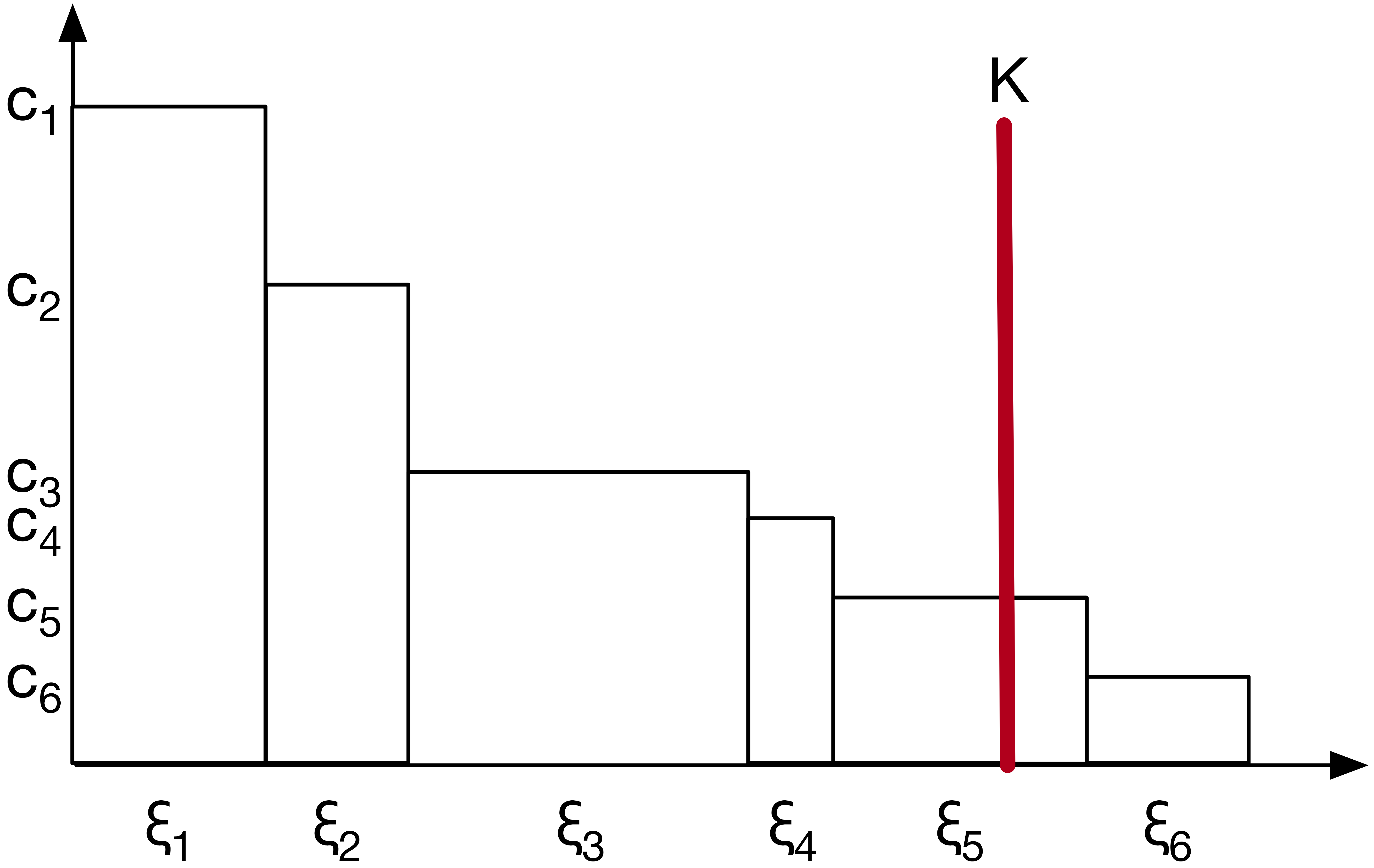
$$\min_{\alpha\geq 0,\gamma\geq 0} \sum_{j\in J} \alpha_{ij}\xi_j x_{ij} + \gamma_i K_i y_i$$

$$\alpha_{ij} + \gamma_i \geq c_{ij} \qquad \forall j \in J$$

and its optimal solution is given by

$$\gamma_i = \begin{cases} \hat{c}_i^{\xi} & y_i^* > 0 \\ 0 & y_i^* = 0 \end{cases} \quad \alpha_{ij} = (c_{ij} - \hat{c}_i^{\xi})^+$$

where $\hat{c}_i^{\xi}$ is the cost of the critical customer where the capacity of the facility is fulfilled (or zero if not).



Dual solution of the subproblem

Dual of the subproblem is given by

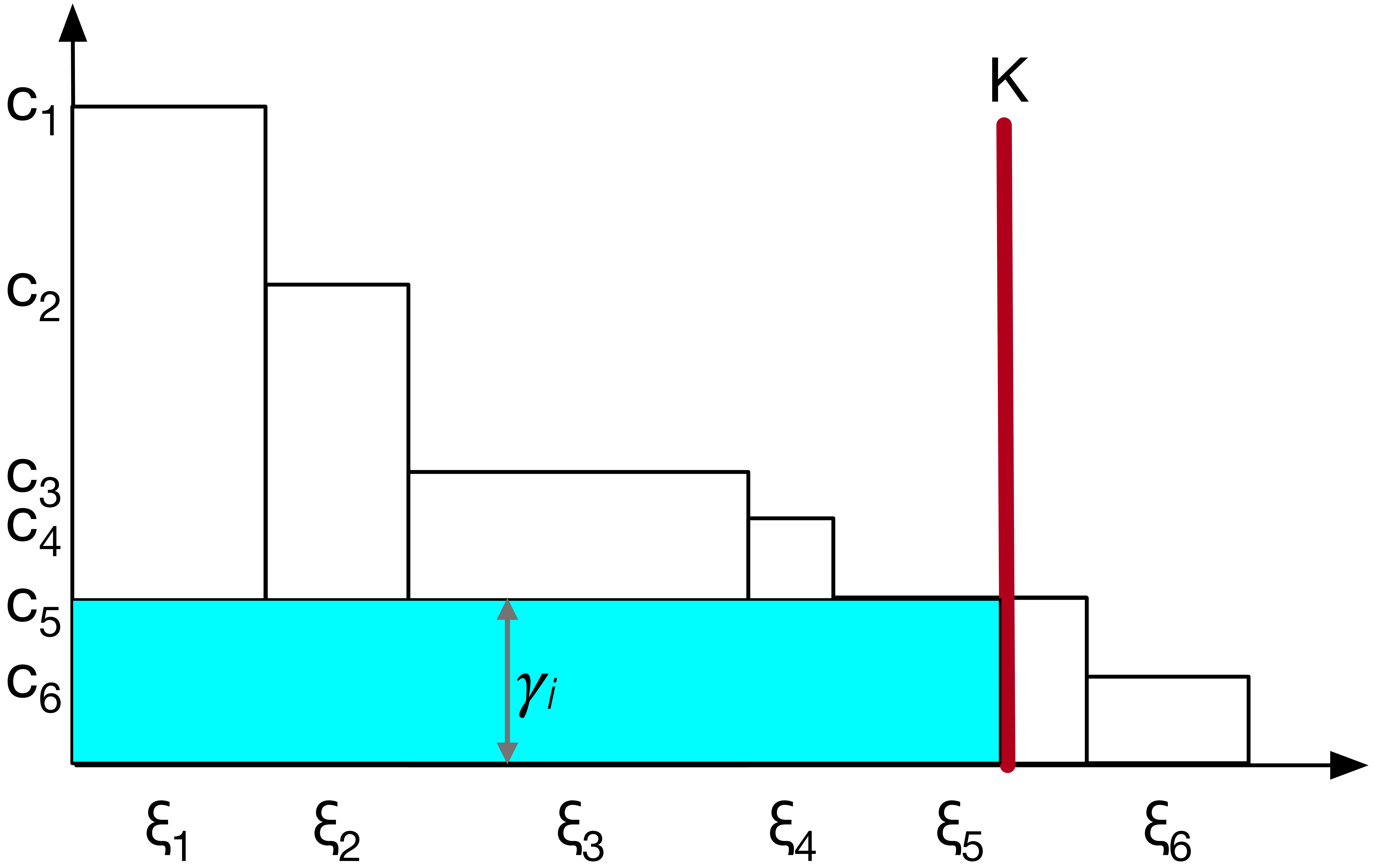
$$\min_{\alpha \geq 0, \gamma \geq 0} \sum_{j \in J} \alpha_{ij} \xi_j x_{ij} + \gamma_i K_i y_i$$

$$\alpha_{ij} + \gamma_i \geq c_{ij} \qquad \forall j \in J$$

and its optimal solution is given by

$$\gamma_i = \begin{cases} \hat{c}_i^{\xi} & y_i^* > 0 \\ 0 & y_i^* = 0 \end{cases} \quad \alpha_{ij} = (c_{ij} - \hat{c}_i^{\xi})^+$$

where $\hat{c}_i^{\xi}$ is the cost of the critical customer where the capacity of the facility is fulfilled (or zero if not).



Dual solution of the subproblem

Dual of the subproblem is given by

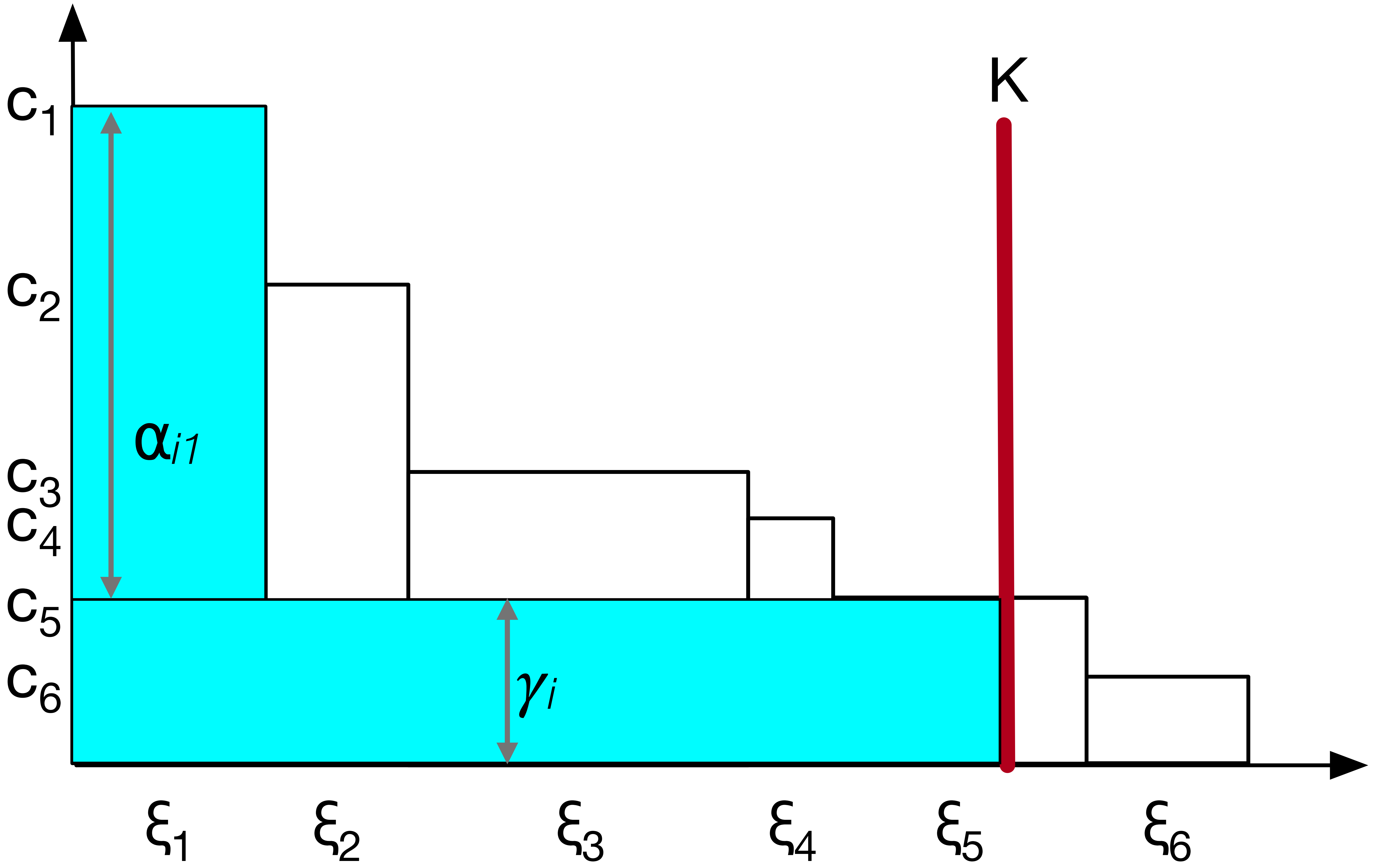
$$\min_{\alpha\geq 0,\gamma\geq 0} \sum_{j\in J} \alpha_{ij}\xi_j x_{ij} + \gamma_i K_i y_i$$

$$\alpha_{ij} + \gamma_i \geq c_{ij} \qquad \forall j \in J$$

and its optimal solution is given by

$$\gamma_i = \begin{cases} \hat{c}_i^{\xi} & y_i^* > 0 \\ 0 & y_i^* = 0 \end{cases} \quad \alpha_{ij} = (c_{ij} - \hat{c}_i^{\xi})^+$$

where $\hat{c}_i^{\xi}$ is the cost of the critical customer where the capacity of the facility is fulfilled (or zero if not).



Google Research

Dual solution of the subproblem

Dual of the subproblem is given by

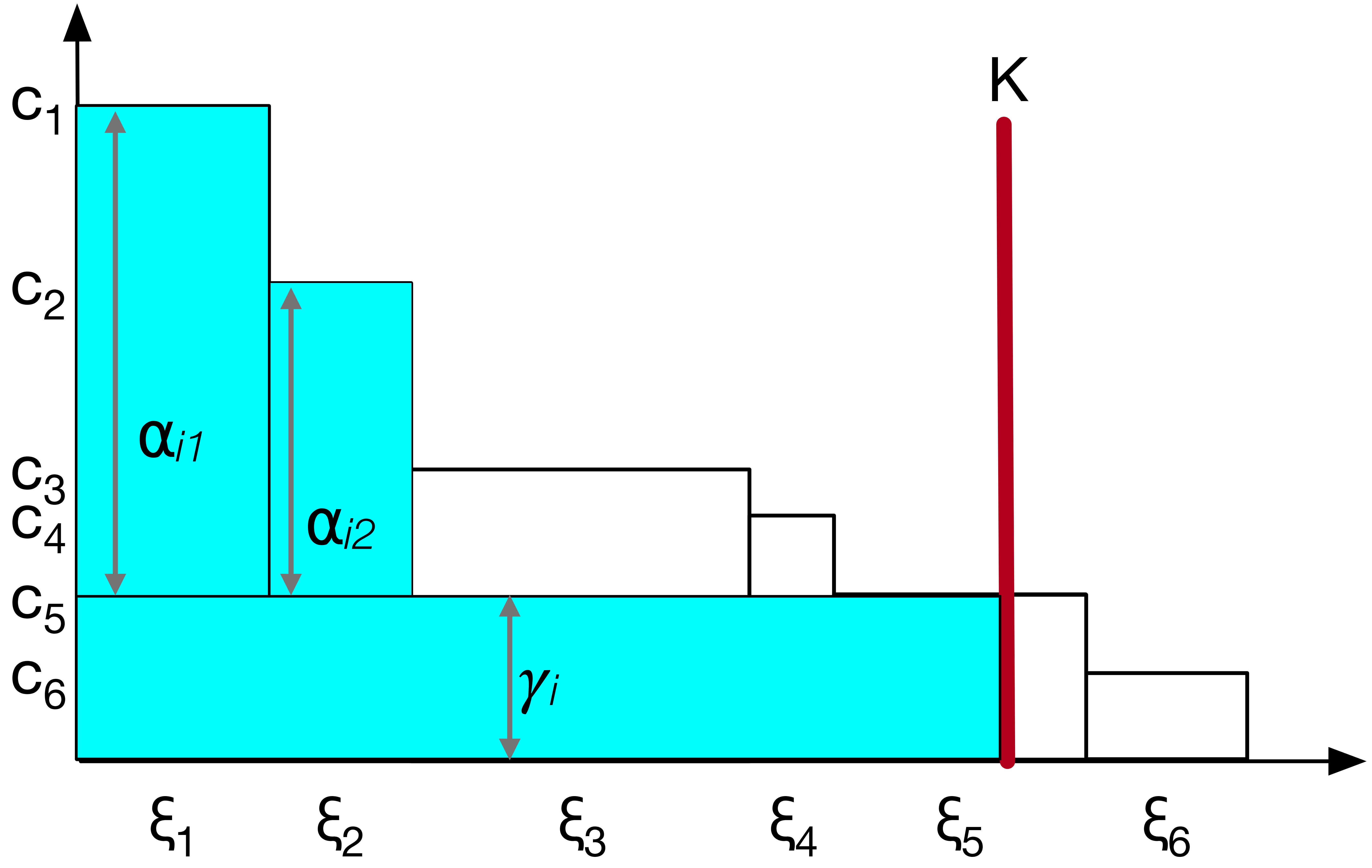
$$\min_{\alpha\geq 0, \gamma \geq 0} \sum_{j\in J} \alpha_{ij}\xi_j x_{ij} + \gamma_i K_i y_i$$

$$\alpha_{ij} + \gamma_i \geq c_{ij} \qquad \forall j \in J$$

and its optimal solution is given by

$$\gamma_i = \begin{cases} \hat{c}_i^{\xi} & y_i^* > 0 \\ 0 & y_i^* = 0 \end{cases} \quad \alpha_{ij} = (c_{ij} - \hat{c}_i^{\xi})^+$$

where $\hat{c}_i^{\xi}$ is the cost of the critical customer where the capacity of the facility is fulfilled (or zero if not).



Dual solution of the subproblem

Dual of the subproblem is given by

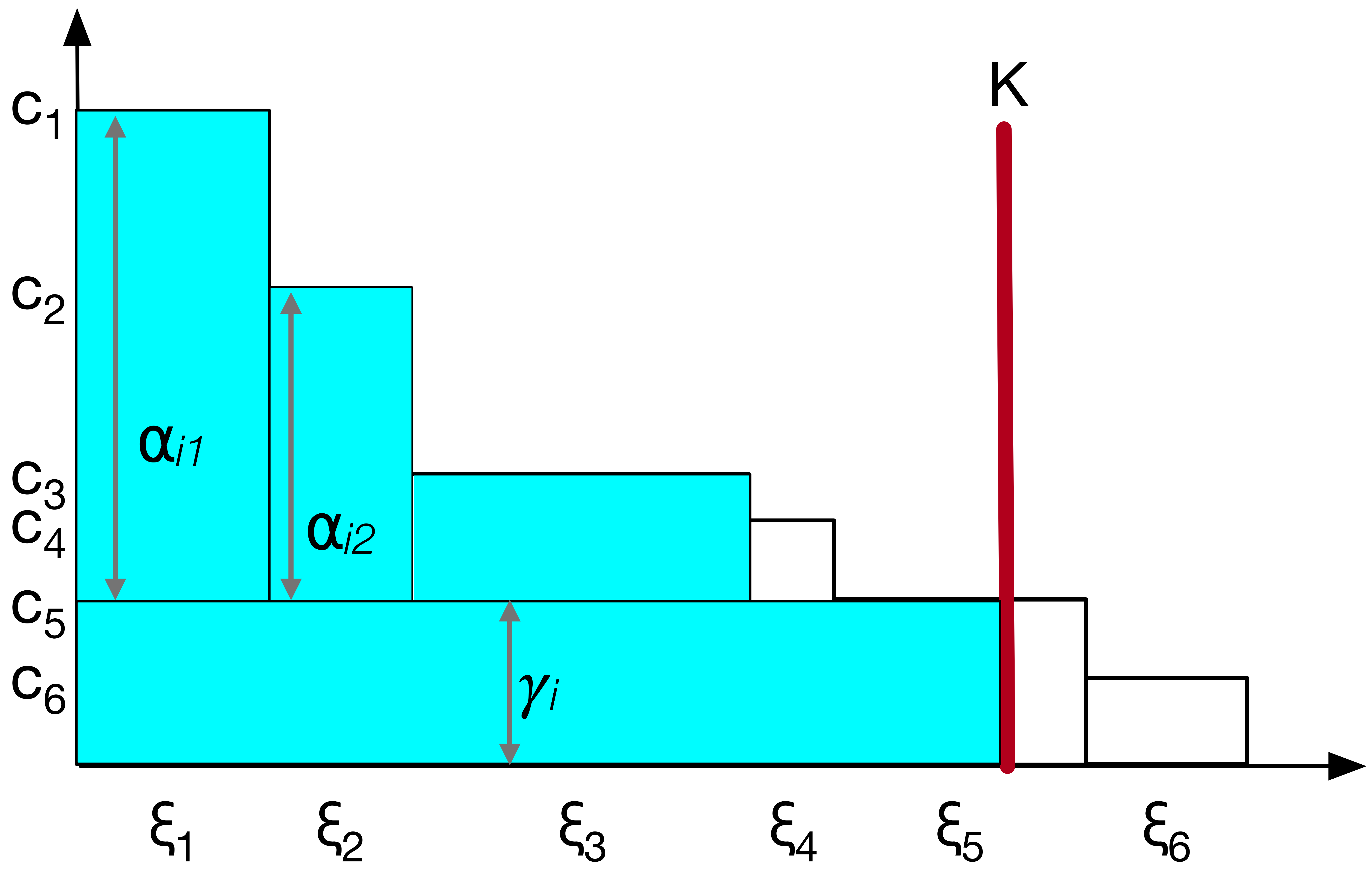
$$\min_{\alpha\geq 0,\gamma\geq 0} \sum_{j\in J} \alpha_{ij}\xi_j x_{ij} + \gamma_i K_i y_i$$

$$\alpha_{ij} + \gamma_i \geq c_{ij} \qquad \forall j \in J$$

and its optimal solution is given by

$$\gamma_i = \begin{cases} \hat{c}_i^\xi & y_i^* > 0 \\ 0 & y_i^* = 0 \end{cases} \quad \alpha_{ij} = (c_{ij} - \hat{c}_i^\xi)^+$$

where $\hat{c}_i^\xi$ is the cost of the critical customer where the capacity of the facility is fulfilled (or zero if not).



Dual solution of the subproblem

Dual of the subproblem is given by

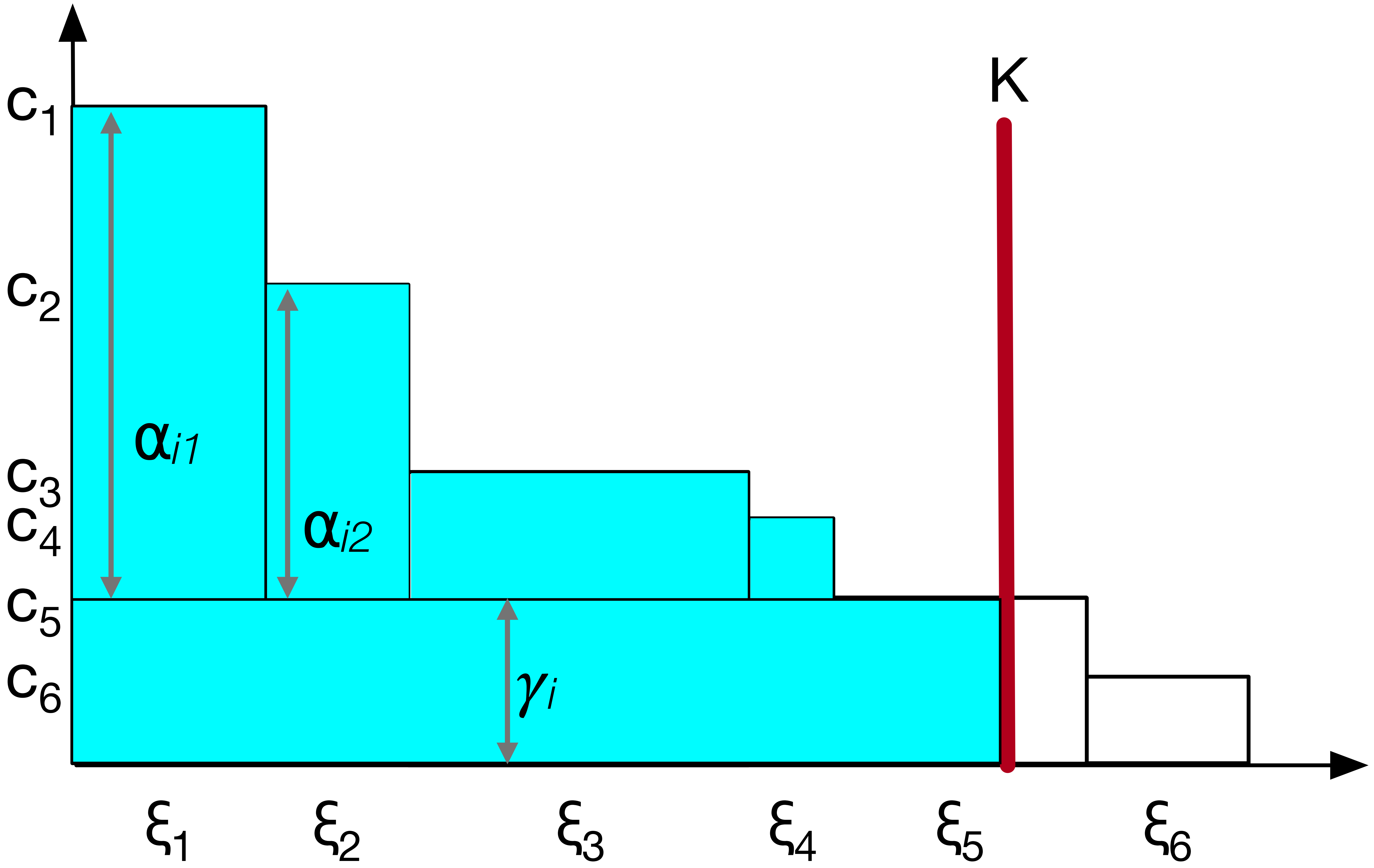
$$\min_{\alpha \geq 0, \gamma \geq 0} \sum_{j \in J} \alpha_{ij} \xi_j x_{ij} + \gamma_i K_i y_i$$

$$\alpha_{ij} + \gamma_i \geq c_{ij} \qquad \forall j \in J$$

and its optimal solution is given by

$$\gamma_i = \begin{cases} \hat{c}_i^\xi & y_i^* > 0 \\ 0 & y_i^* = 0 \end{cases} \quad \alpha_{ij} = (c_{ij} - \hat{c}_i^\xi)^+$$

where $\hat{c}_i^\xi$ is the cost of the critical customer where the capacity of the facility is fulfilled (or zero if not).



Benders formulation for a discrete set of scenarios

Given a discrete set of scenarios $s \in S$ with probability p_s, we can reformulate

$$\min_{x,y} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} h_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{s \in S} p_s \theta_s^i$$

$$(x, y) \in \mathscr{X} \qquad x, y \in \{0,1\}$$

$$\theta_s^i \leq \hat{Q}^i(x, y, \xi^s) \qquad \forall i \in I, s \in S$$

where

$$\hat{Q}^i(x, y, \xi^s) := \min_{\alpha \geq 0, \gamma \geq 0} \left\{ \sum_{j \in J} \alpha_{ij} \xi_j x_{ij} + \gamma_i K_i y_i : \alpha_{ij} + \gamma_i \geq c_{ij} \forall j \in J \right\}$$

Benders formulation for a discrete set of scenarios

Given a discrete set of scenarios $s \in S$ with probability p_s, we can reformulate

$$\min_{x,y} \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{j\in J} h_{ij}\mu_j x_{ij} - \sum_{i\in I}\sum_{s\in S} p_s \theta_s^i$$

$$(x,y) \in \mathcal{X} \qquad x, y \in \{0,1\}$$

$$\theta_s^i \leq \hat{Q}^i(x,y,\xi^s) \qquad \forall i \in I, s \in S$$

and $\hat{Q}^i(x,y,\xi^s) \leq \sum_{j\in J} \alpha_{ij}^* \xi_j x_{ij} + \gamma_i^* K_i y_i$ for any feasible dual solution (α^*, γ^*)

Benders formulation for a discrete set of scenarios

Given a discrete set of scenarios $s \in S$ with probability p_s, we can reformulate

$$\min_{x,y} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} h_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{s \in S} p_s \theta_s^i$$

$$(x, y) \in \mathcal{X} \qquad x, y \in \{0,1\}$$

$$\theta_s^i \leq \hat{Q}^i(x, y, \xi^s) \qquad \forall i \in I, s \in S$$

and $\hat{Q}^i(x, y, \xi^s) \leq \sum_{j \in J} \alpha_{ij}^* \xi_j x_{ij} + \gamma_i^* K_i y_i$ for any feasible dual solution (α^*, γ^*)

in particular $\hat{Q}^i(x, y, \xi^s) \leq \sum_{j \in J} \left(c_{ij} - \hat{c}_i^{\xi,*} \right)^+ \cdot \xi_j \cdot x_{ij} + \hat{c}_i^{\xi,*} \cdot K_i \cdot y_i$

where $\hat{c}_i^{\xi,*}$ is computed as before from any valid assignment.

Benders method for discrete scenarios

1.- Solve main problem $\min_{x,y} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} h_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{s \in S} p_s \theta_s^i$

$$(x, y) \in \mathcal{X} \quad x, y \in \{0,1\}$$

2.- Given incumbent solution (x^k, y^k) compute the cost of critical customer $\hat{c}_i^{\xi^s,k}$ for each open facility i and for each scenario ξ^s.

3.- Add the Bender optimality cuts to the main problem

4.- Resolve main problem and iterate.

Google Research

Benders method for discrete scenarios

1.- Solve main problem $$\min_{x,y} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} h_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{s \in S} p_s \theta_s^i$$

$$(x, y) \in \mathscr{X} \quad x, y \in \{0,1\}$$

2.- Given incumbent solution (x^k, y^k) compute the cost of critical customer $\hat{c}_i^{\xi^s,k}$ for each open facility i and for each scenario ξ^s.

3.- Add the Bender optimality cuts to the main problem

$$\theta_s^i \leq \sum_{j \in J} \left(c_{ij} - \hat{c}_i^{\xi^s,k} \right)^+ \cdot \xi_j^s \cdot x_{ij} + \hat{c}_i^{\xi^s,k} \cdot K_i \cdot y_i$$

4.- Resolve main problem and iterate.

Stochastic facility location problems with outsourcing costs

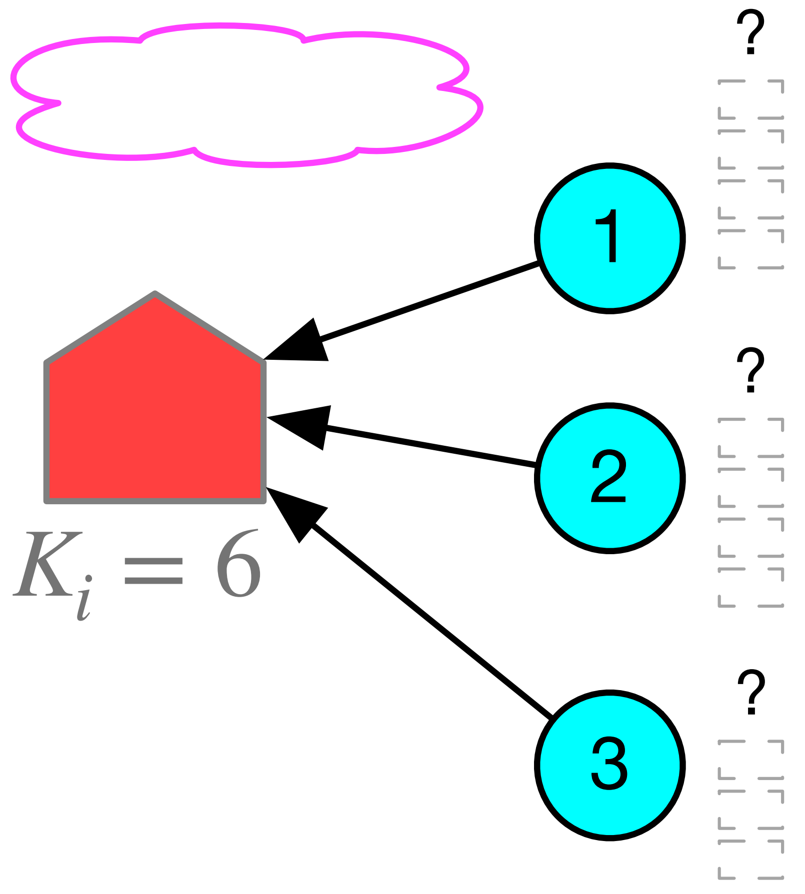
1. Bender formulation for a discrete set of scenarios (for example, a sample average approximation of the demand distributions)

2. **Bender formulation for general distributions.**

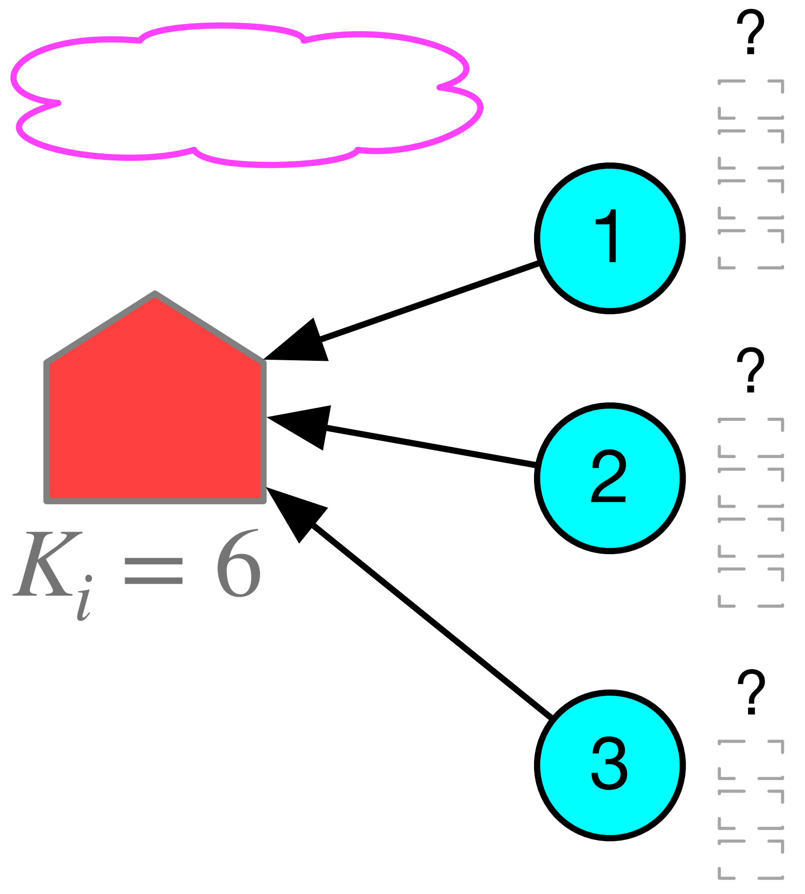
3. Strengthened formulation.

4. Computational experiments

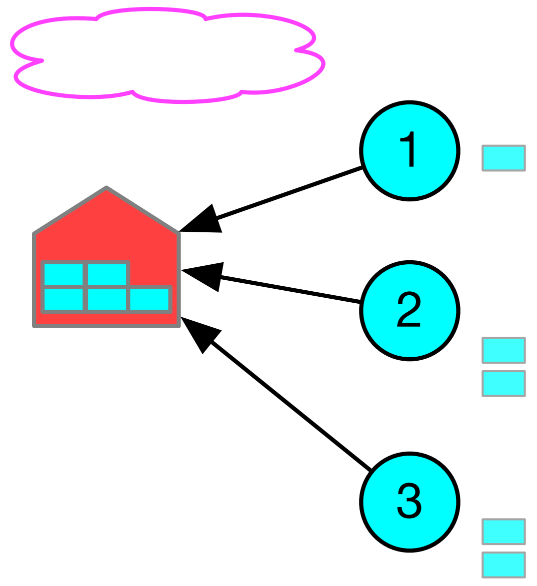
Dual solution of the subproblem



Dual solution of the subproblem



Scenario 1



Primal solution

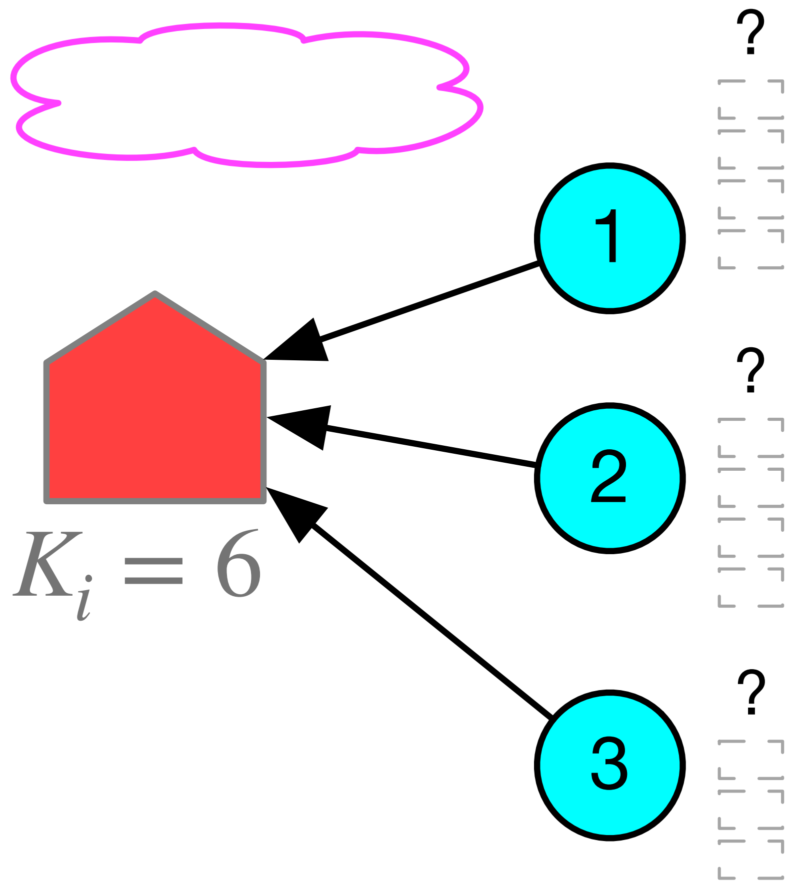
$$w_{i1} = 1$$
$$w_{i2} = 2$$
$$w_{i3} = 2$$

Dual solution

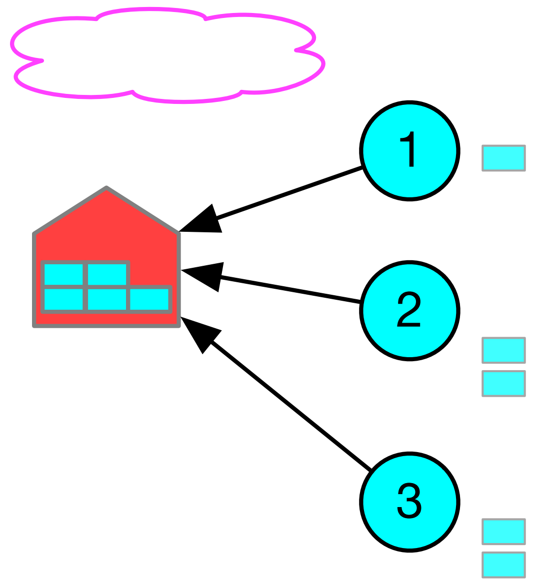
$$\tau^i = \varnothing \Rightarrow \hat{v}_i = 0$$
$$\alpha_{i1} = g_{i1} - c_{i1}$$
$$\alpha_{i2} = g_{i2} - c_{i2}$$
$$\alpha_{i3} = g_{i3} - c_{i3}$$

Google Research

Dual solution of the subproblem

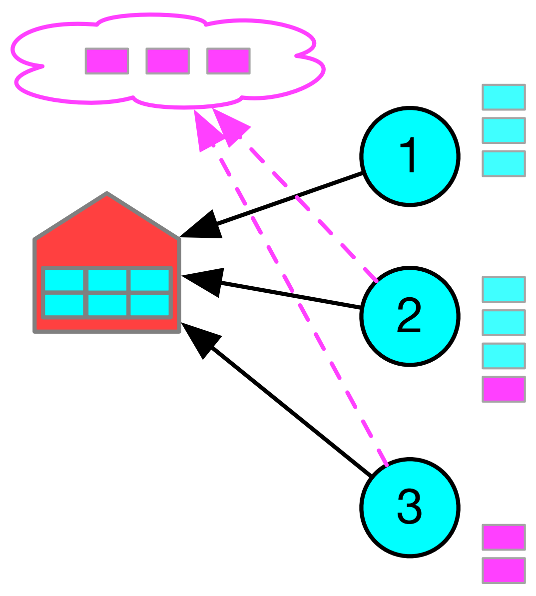


Scenario 1



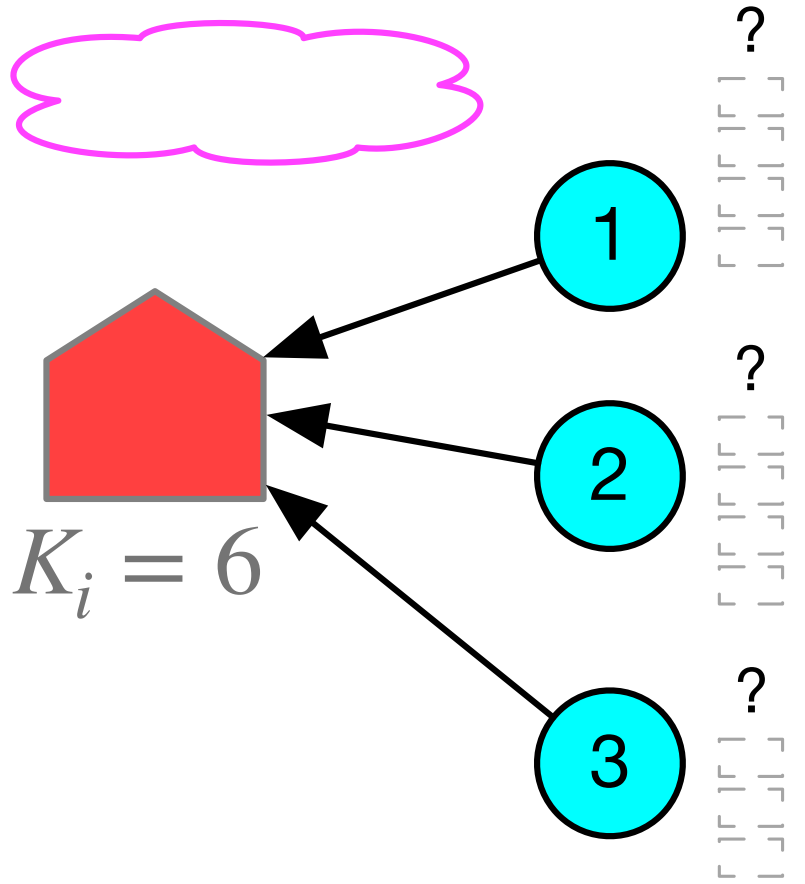
	Primal solution	Dual solution
Scenario 1	$w_{i1} = 1$	$\tau^i = \emptyset \Rightarrow \hat{v}_i = 0$
	$w_{i2} = 2$	$\alpha_{i1} = g_{i1} - c_{i1}$
	$w_{i3} = 2$	$\alpha_{i2} = g_{i2} - c_{i2}$
		$\alpha_{i3} = g_{i3} - c_{i3}$

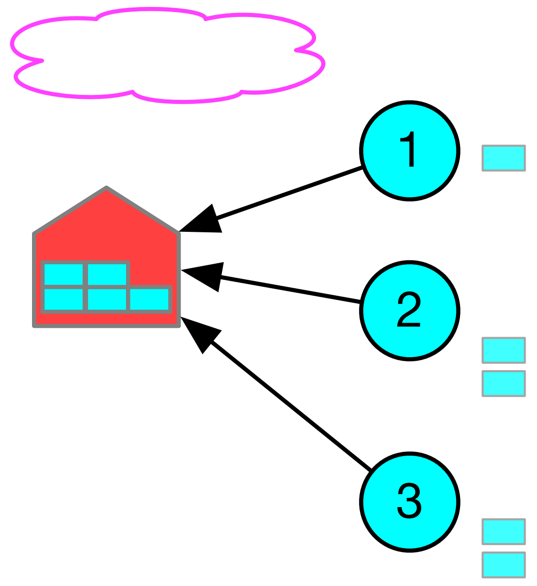
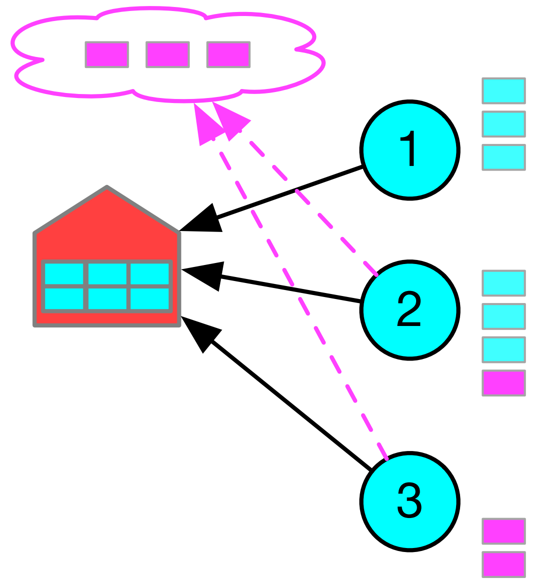
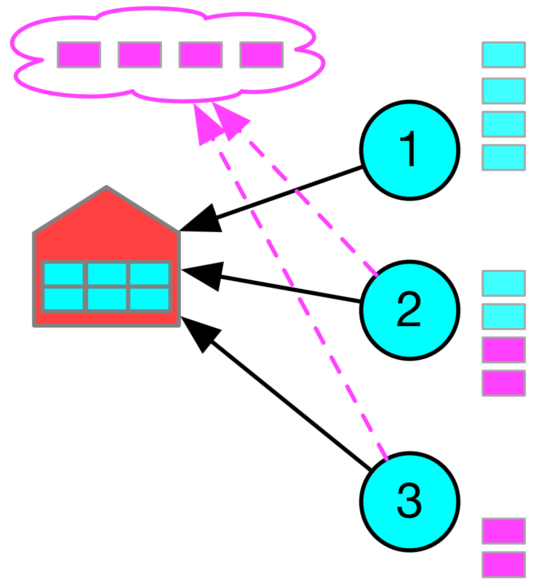
Scenario 2



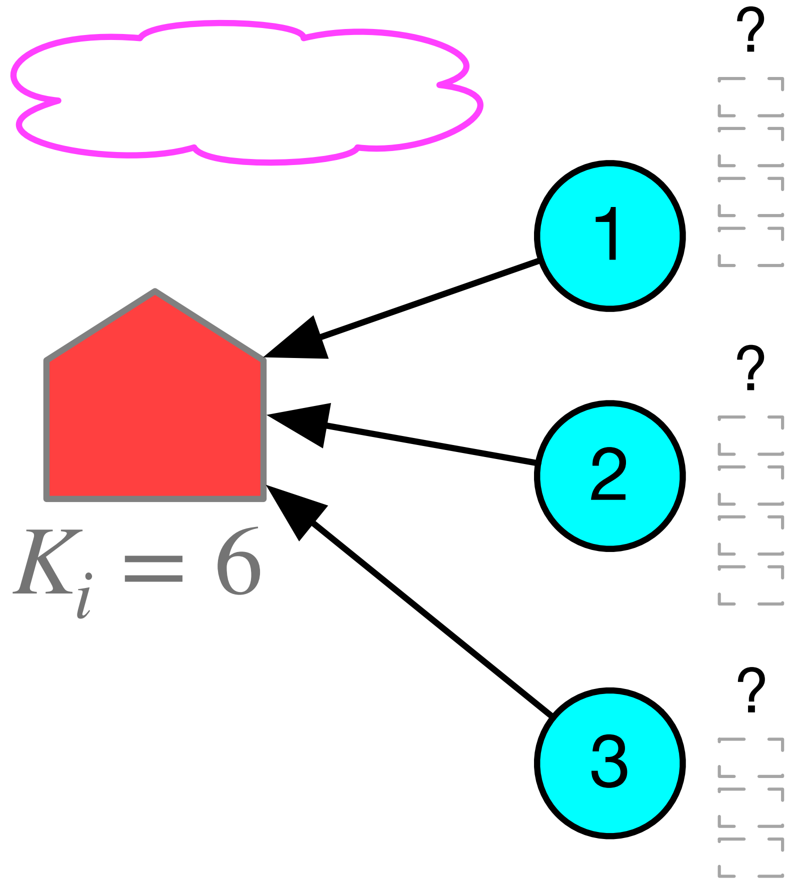
	Primal solution	Dual solution
Scenario 2		$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$
	$w_{i1} = 3$	$\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$
	$w_{i2} = 3$	$\alpha_{i2} = 0$
	$w_{i3} = 0$	$\alpha_{i3} = 0$

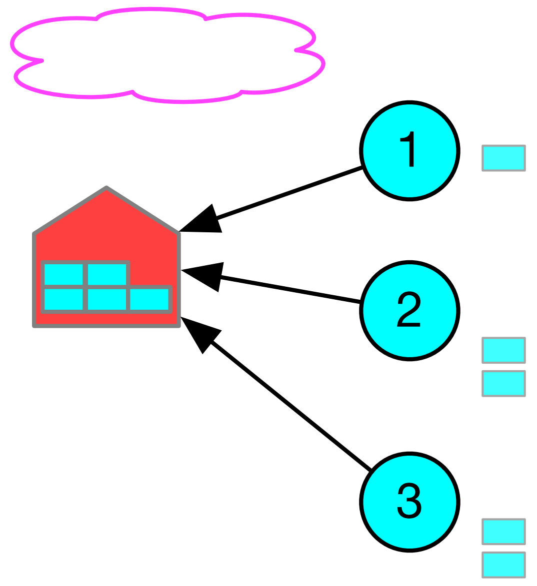
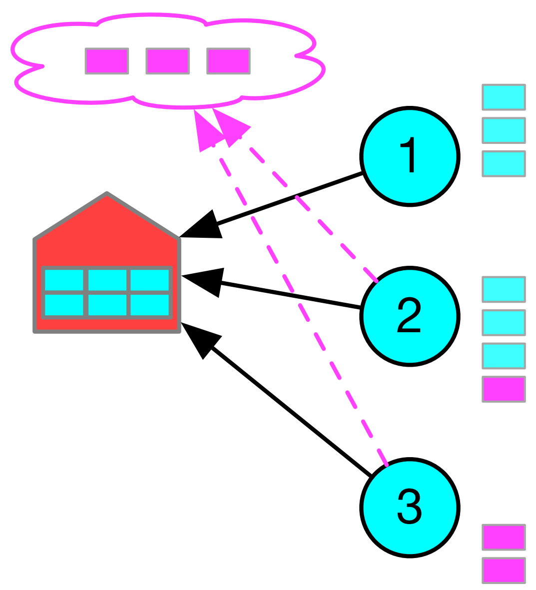
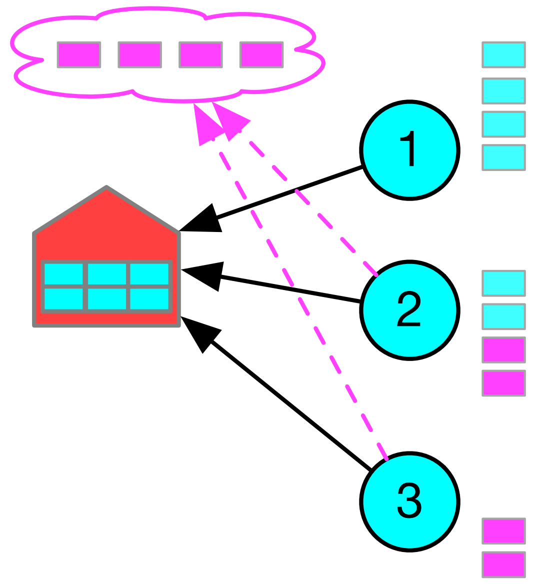
Dual solution of the subproblem



		Primal solution	Dual solution
Scenario 1		$w_{i1} = 1$ $w_{i2} = 2$ $w_{i3} = 2$	$\tau^i = \varnothing \Rightarrow \hat{v}_i = 0$ $\alpha_{i1} = g_{i1} - c_{i1}$ $\alpha_{i2} = g_{i2} - c_{i2}$ $\alpha_{i3} = g_{i3} - c_{i3}$
Scenario 2		$w_{i1} = 3$ $w_{i2} = 3$ $w_{i3} = 0$	$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$ $\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$ $\alpha_{i2} = 0$ $\alpha_{i3} = 0$
Scenario 3		$w_{i1} = 4$ $w_{i2} = 2$ $w_{i3} = 0$	$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$ $\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$ $\alpha_{i2} = 0$ $\alpha_{i3} = 0$

Computing $\mathbb{E}\left[Q^i(x, y, \xi)\right]$



		Primal solution	Dual solution
Scenario 1		$w_{i1} = 1$ $w_{i2} = 2$ $w_{i3} = 2$	$\tau^i = \varnothing \Rightarrow \hat{v}_i = 0$ $\alpha_{i1} = g_{i1} - c_{i1}$ $\alpha_{i2} = g_{i2} - c_{i2}$ $\alpha_{i3} = g_{i3} - c_{i3}$
Scenario 2		$w_{i1} = 3$ $w_{i2} = 3$ $w_{i3} = 0$	$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$ $\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$ $\alpha_{i2} = 0$ $\alpha_{i3} = 0$
Scenario 3		$w_{i1} = 4$ $w_{i2} = 2$ $w_{i3} = 0$	$\tau^i = 2 \Rightarrow \hat{v}_i = g_{i2} - c_{i2}$ $\alpha_{i1} = g_{i1} - c_{i1} - \hat{v}_i$ $\alpha_{i2} = 0$ $\alpha_{i3} = 0$

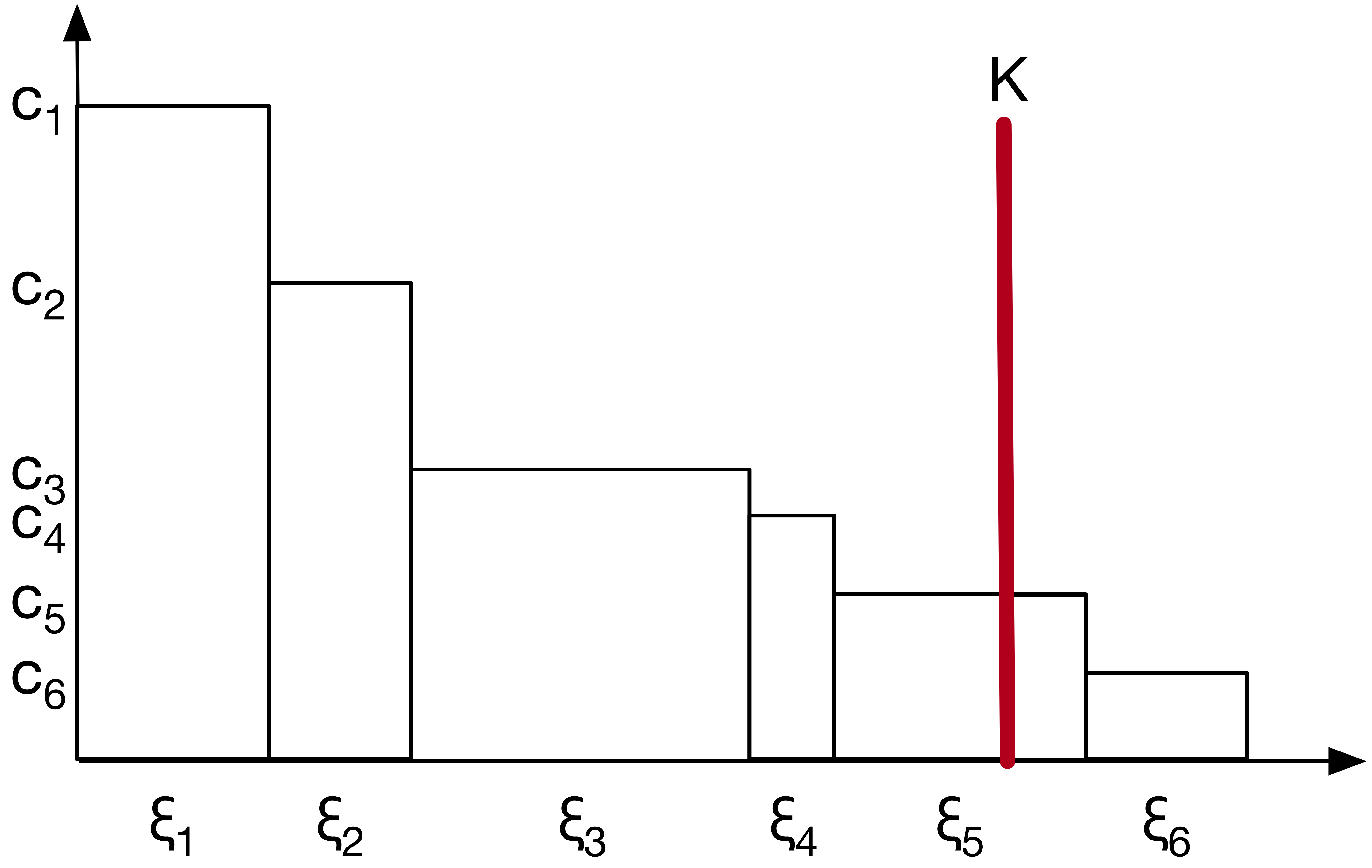
Revisiting the dual problem

Let $S_j^i(x,\xi)$ be the aggregated demand of the best j customers assigned to i:

$$S_j^i(x,\xi) = \sum_{l \le j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x,y,\xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S_j^i(x,\xi), K_i\}$$



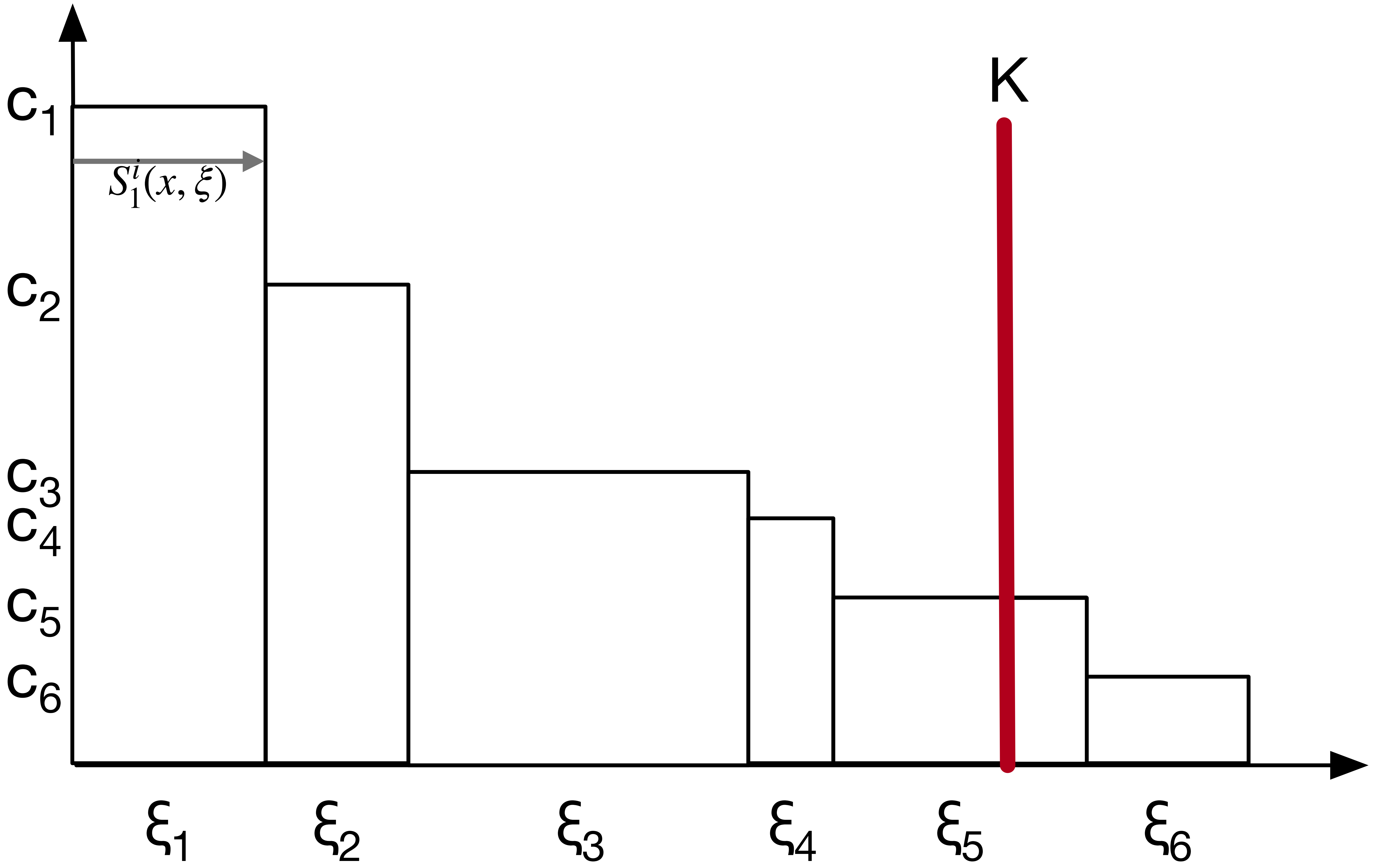
Revisiting the dual problem

Let $S^i_j(x,\xi)$ be the aggregated demand of the best j customers assigned to i:

$$S^i_j(x,\xi) = \sum_{l \le j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x,y,\xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S^i_j(x,\xi), K_i\}$$



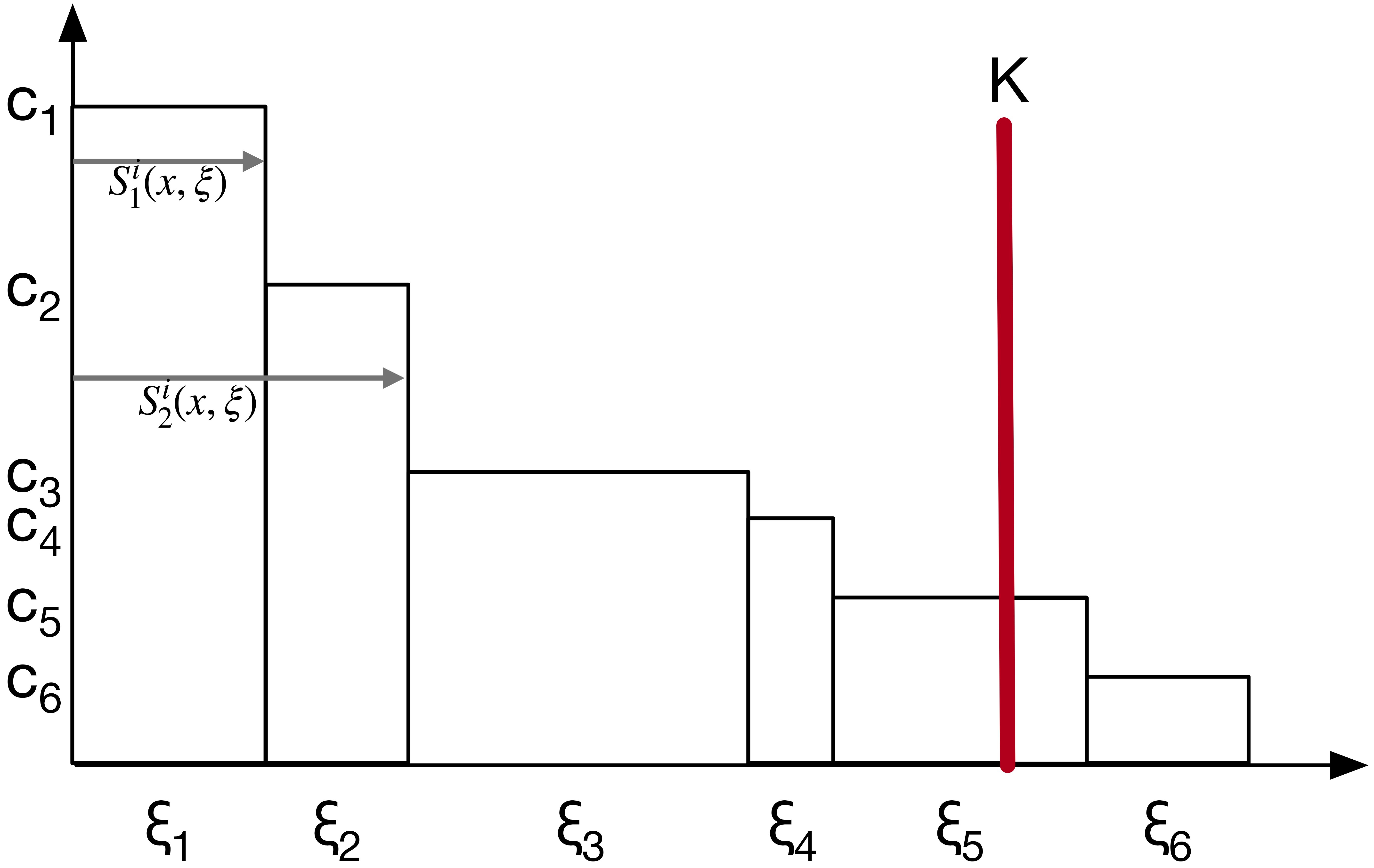
Revisiting the dual problem

Let $S^i_j(x, \xi)$ be the aggregated demand of the best j customers assigned to i:

$$S^i_j(x, \xi) = \sum_{l \leq j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x, y, \xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S^i_j(x, \xi), K_i\}$$



Google Research

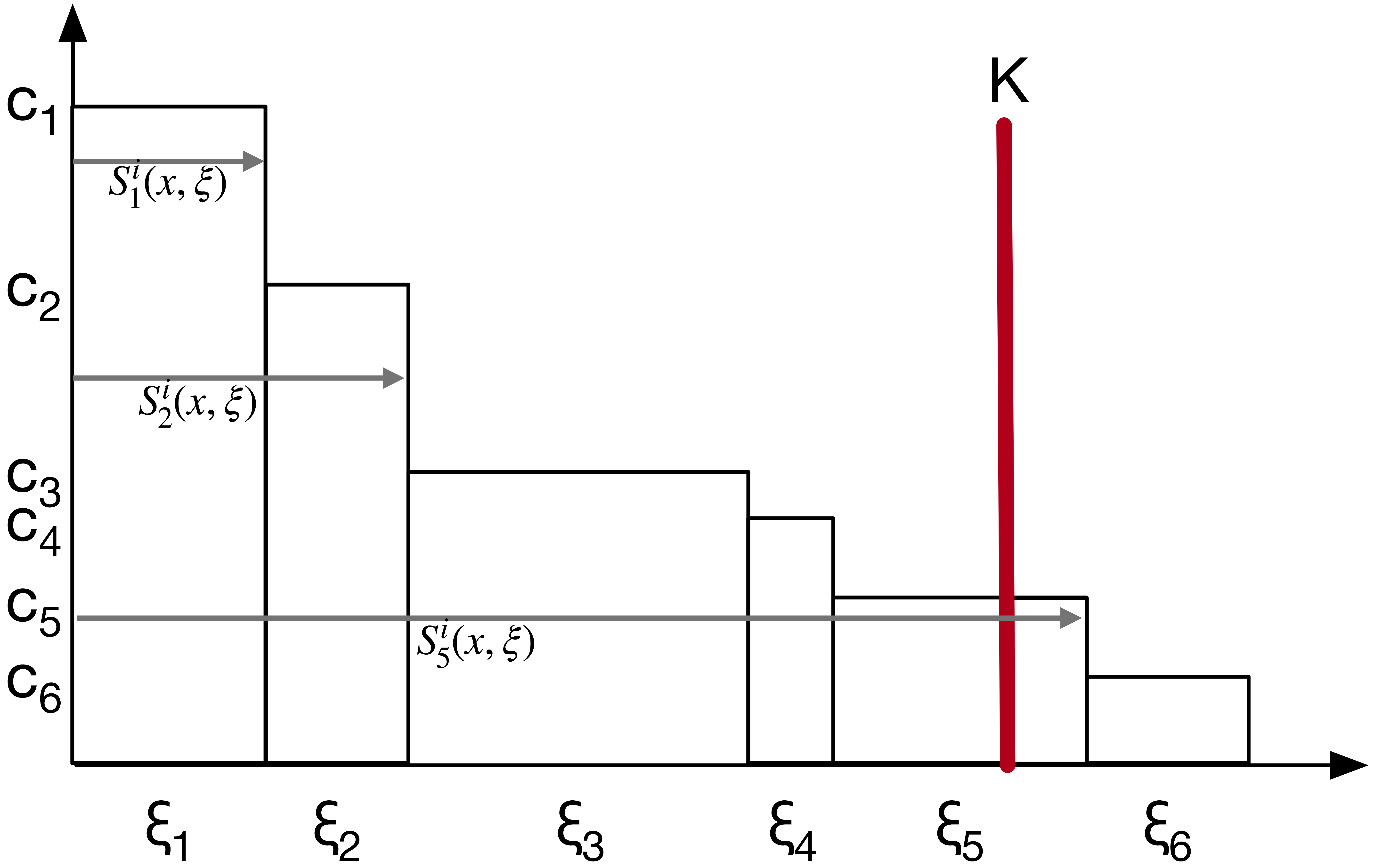
Revisiting the dual problem

Let $S_j^i(x, \xi)$ be the aggregated demand of the best j customers assigned to i:

$$S_j^i(x, \xi) = \sum_{l \leq j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x, y, \xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S_j^i(x, \xi), K_i\}$$



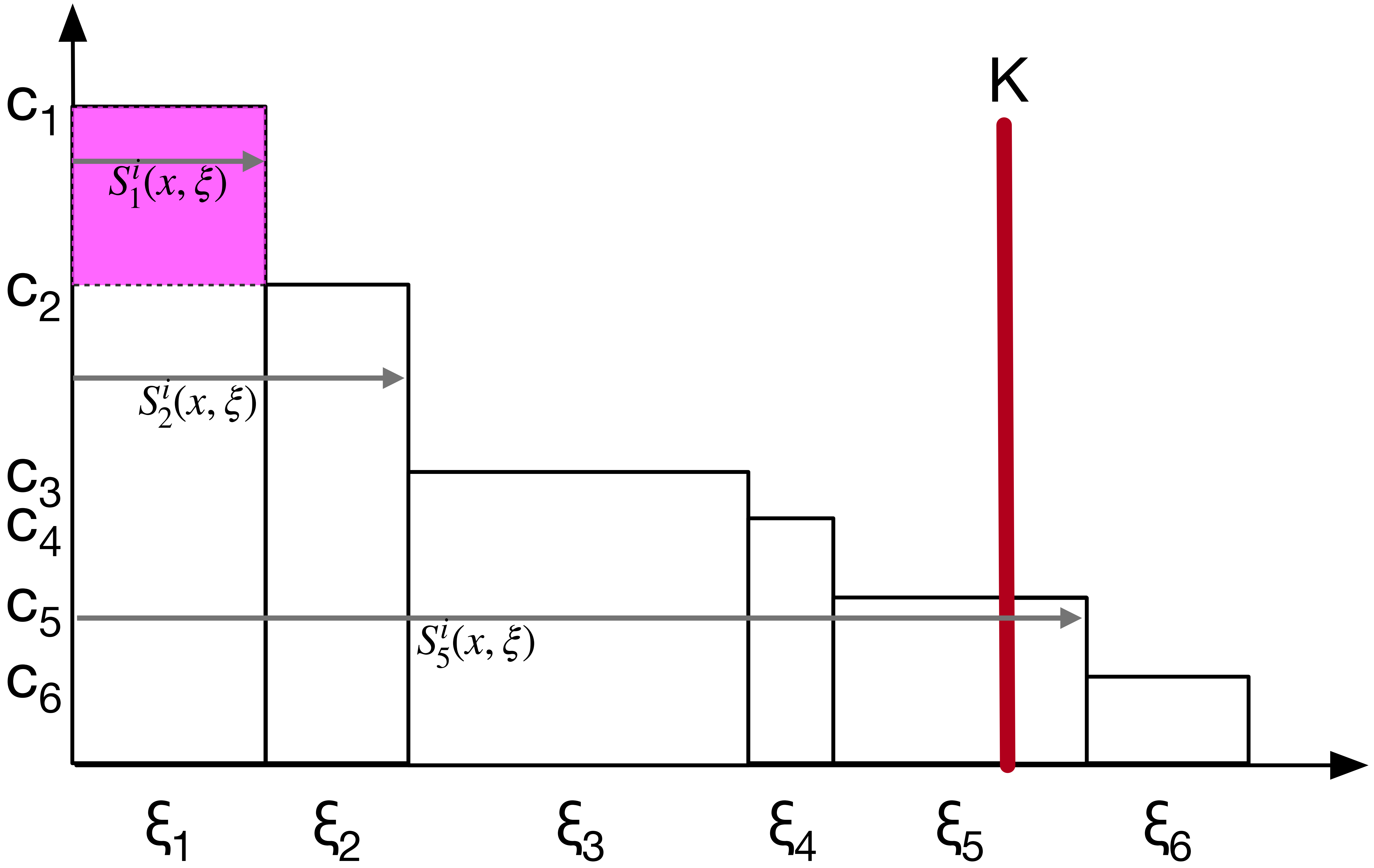
Revisiting the dual problem

Let $S_j^i(x, \xi)$ be the aggregated demand of the best j customers assigned to i:

$$S_j^i(x, \xi) = \sum_{l \le j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x, y, \xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S_j^i(x, \xi), K_i\}$$



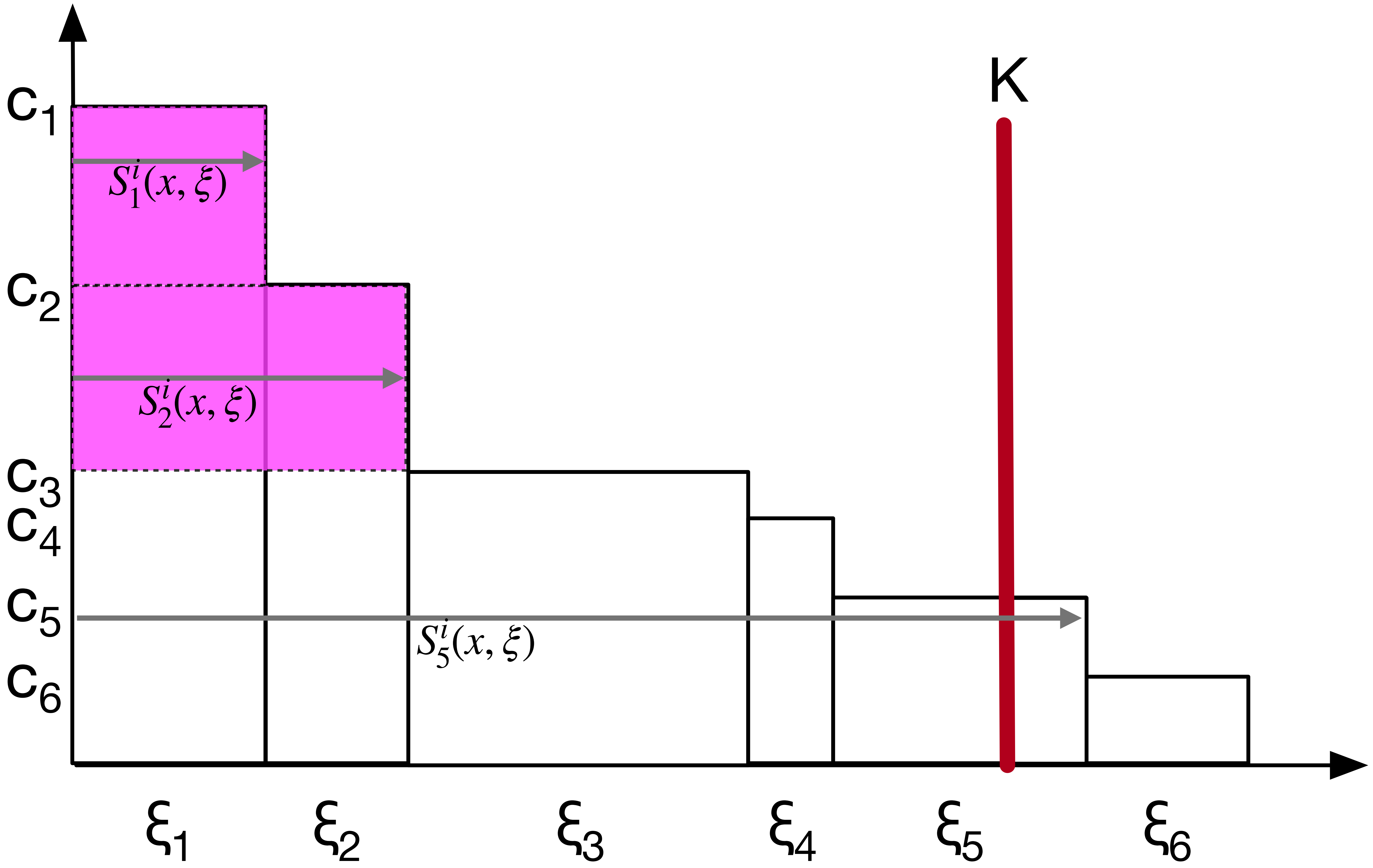
Revisiting the dual problem

Let $S_j^i(x, \xi)$ be the aggregated demand of the best j customers assigned to i:

$$S_j^i(x, \xi) = \sum_{l \leq j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x, y, \xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S_j^i(x, \xi), K_i\}$$



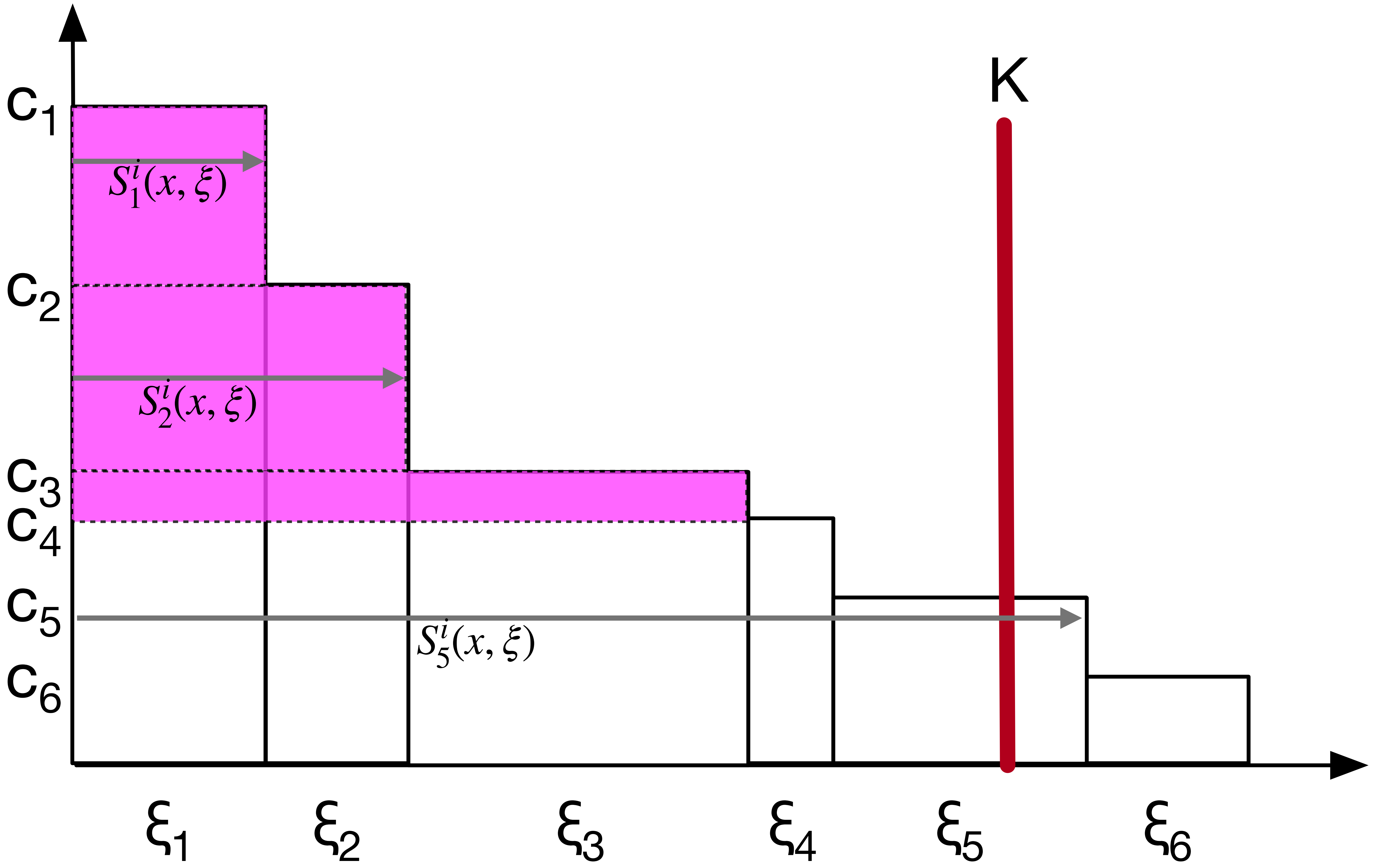
Revisiting the dual problem

Let $S_j^i(x, \xi)$ be the aggregated demand of the best j customers assigned to i:

$$S_j^i(x, \xi) = \sum_{l \leq j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x, y, \xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S_j^i(x, \xi), K_i\}$$



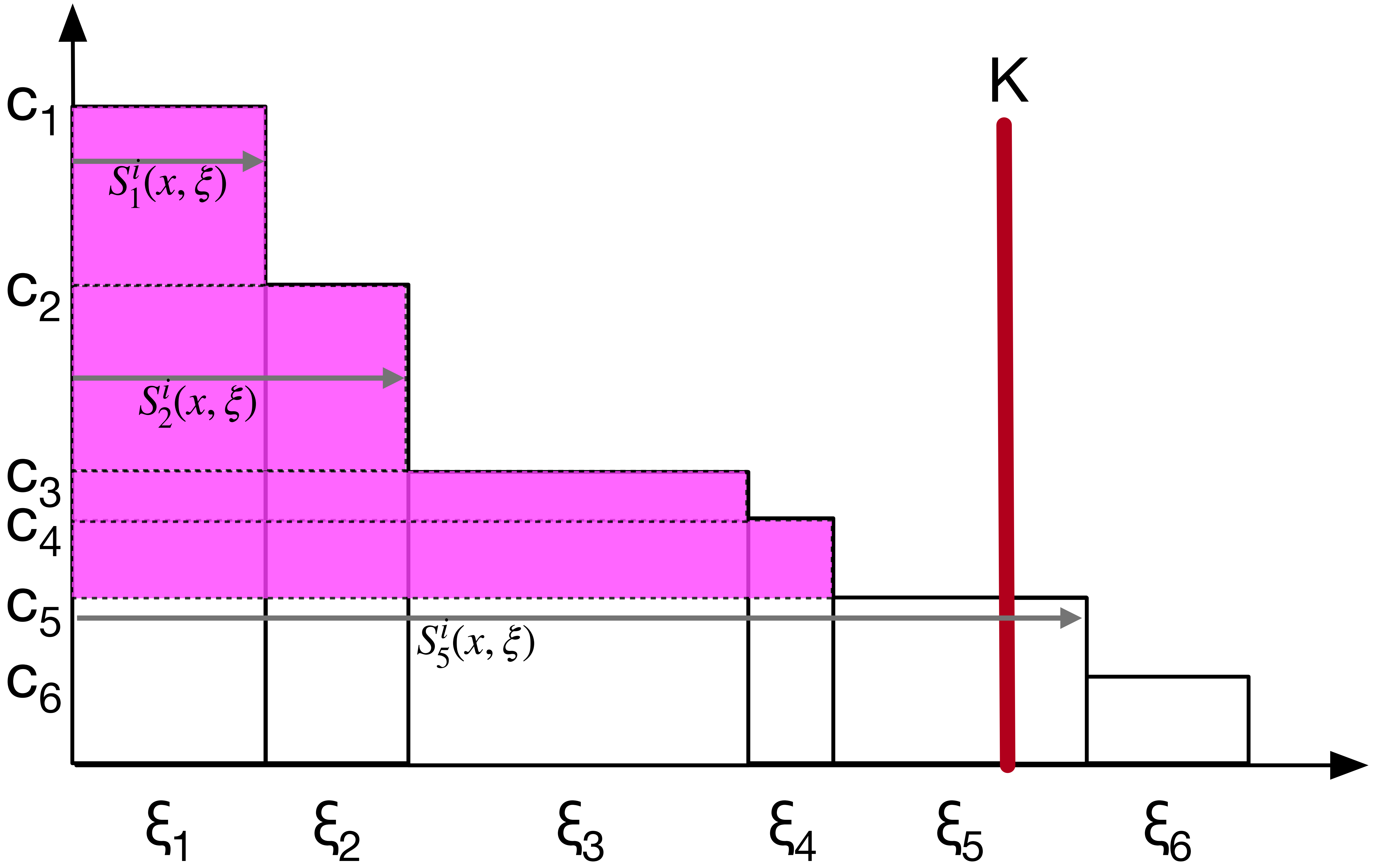
Revisiting the dual problem

Let $S^i_j(x, \xi)$ be the aggregated demand of the best j customers assigned to i:

$$S^i_j(x, \xi) = \sum_{l \le j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x, y, \xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S^i_j(x, \xi), K_i\}$$



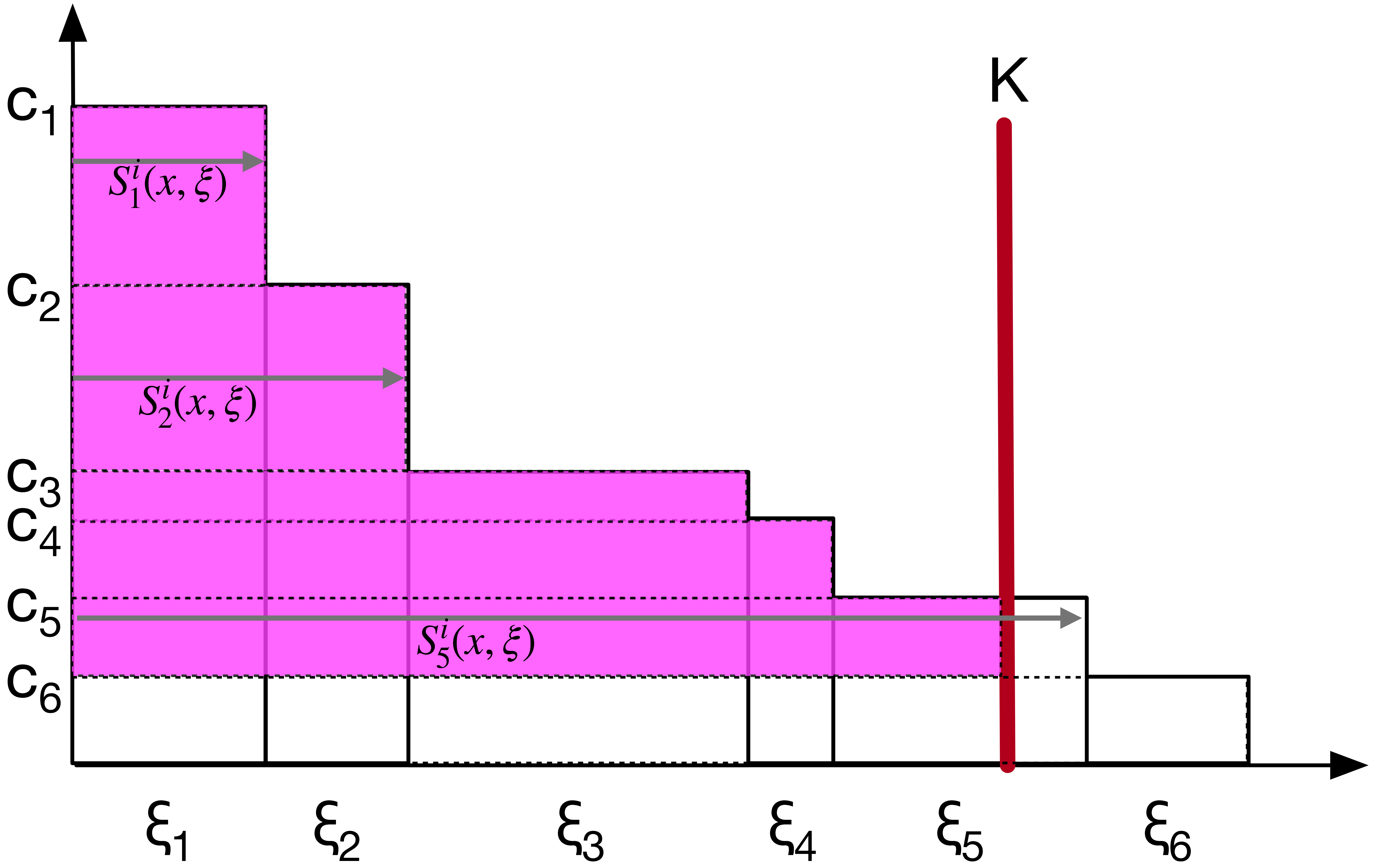
Revisiting the dual problem

Let $S_j^i(x, \xi)$ be the aggregated demand of the best j customers assigned to i:

$$S_j^i(x, \xi) = \sum_{l \le j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x, y, \xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S_j^i(x, \xi), K_i\}$$



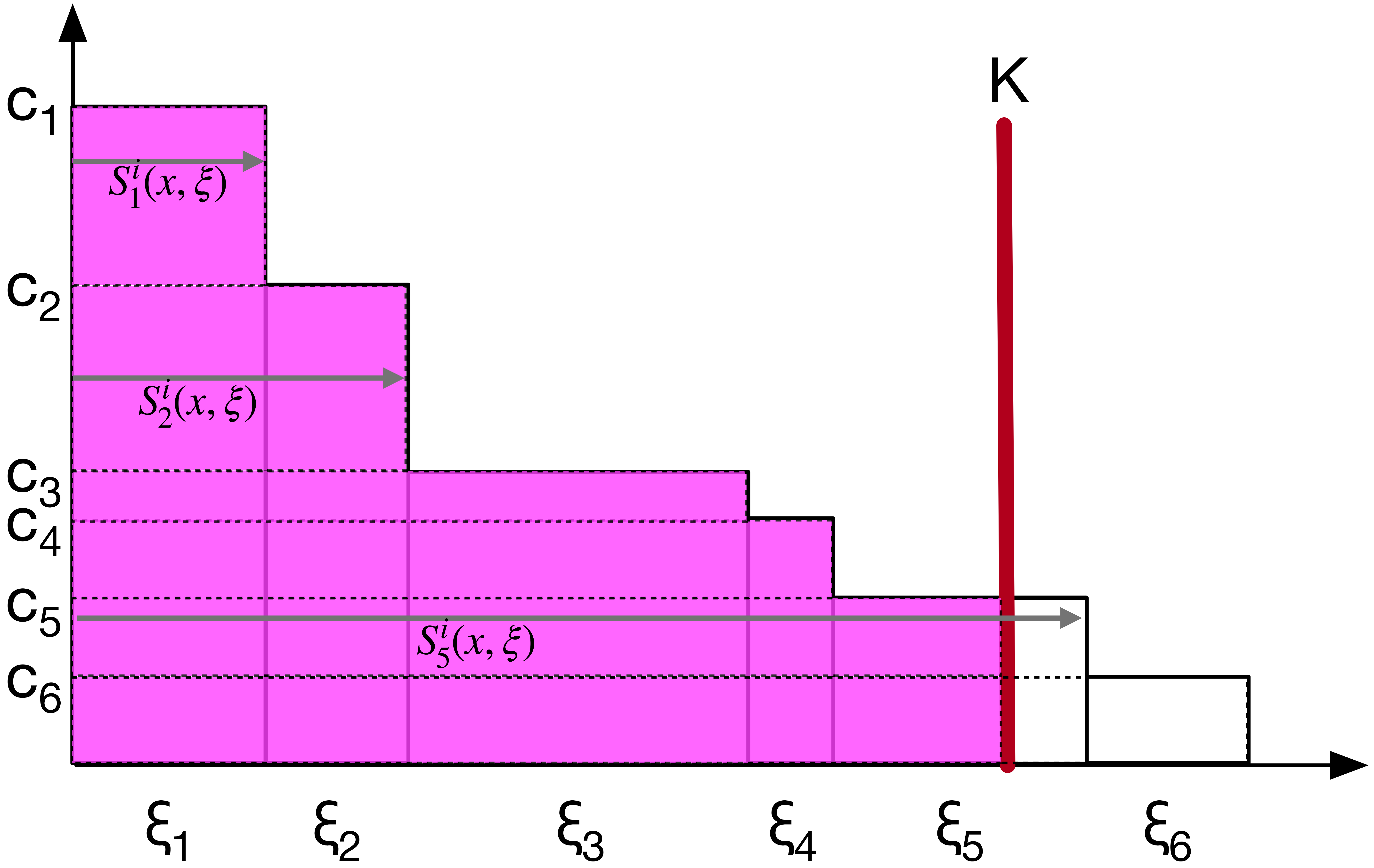
Revisiting the dual problem

Let $S_j^i(x, \xi)$ be the aggregated demand of the best j customers assigned to i:

$$S_j^i(x, \xi) = \sum_{l \leq j} \xi_l x_{il}$$

Then the optimal value of the subproblem is given by

$$Q^i(x, y, \xi) = \sum_{j \in J} (c_{ij} - c_{i,j+1}) \cdot \min\{S_j^i(x, \xi), K_i\}$$



Expected value of the second-stage problem

For a fixed x we can obtain closed formulas for many demand distributions:

- Bernoulli distribution with mean μ_j: $\mathbb{E}[w_{ij}^{\xi}] = \mu_j \cdot F_{S_{j-1}}(K_i - 1)$
- Poisson distribution with mean μ_j:

$$\mathbb{E}[\min\{S_j(x,\xi), K_i\}] = K_i \cdot (1 - f_{Poisson(\mu_{S_j(x,\xi)})}(K_i)) + (\mu_{S_j(x,\xi)} - K_i) \cdot F_{Poisson(\mu_{S_j(x,\xi)})}(K_i - 1)$$

- Exponential distribution with mean μ:

$$\mathbb{E}[\min\{S_j(x,\xi), K_i\}] = j \cdot \mu \cdot F_{Gamma(j+1,1/\mu)}(K_i) + K_i\left(1 - F_{Gamma(j,1/\mu)}(K_i)\right)$$

Google Research

Benders formulation for a discrete set of scenarios

Main problem

$$\min_{x,y} \left\{ \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} h_{ij} \mu_j x_{ij} - \sum_{i \in I} \mathbb{E}\left[Q^i(x, y, \xi)\right] : (x, y) \in \mathcal{X} \right\}$$

$$\min_{x,y} \left\{ \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} h_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{j \in J} (c_{i,\sigma^i(j)} - c_{i,\sigma^i(j+1)}) \cdot \mathbb{E}\left[\min\{S^i_j(x, \xi), K_i\}\right] : (x, y) \in \mathcal{X} \right\}$$

Google Research

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

By the law of total probabilities:

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi)\leq K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi)> K_i}\right]$$

$$S_j^i(x,\xi) = \sum_{l\leq j} \xi_l x_{il}$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

By the law of total probabilities:

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi)\le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi)>K_i}\right]$$

$$= \mathbb{E}\left[S_j^i(x,\xi) \cdot 1_{S_j^i(x,\xi)\le K_i}\right] + \mathbb{E}\left[K_i \cdot 1_{S_j^i(x,\xi)>K_i}\right]$$

$$S_j^i(x,\xi) = \sum_{l\le j} \xi_l x_{il}$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

Google Research

By the law of total probabilities:

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi) \le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi) > K_i}\right]$$

$$= \mathbb{E}\left[S_j^i(x,\xi) \cdot 1_{S_j^i(x,\xi) \le K_i}\right] + \mathbb{E}\left[K_i \cdot 1_{S_j^i(x,\xi) > K_i}\right]$$

$$= \sum_{l \le j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x,\xi) \le K_i}\right] \cdot x_{il} + K_i \cdot \mathbb{P}\left[S_j^i(x,\xi) > K_i\right]$$

$$S_j^i(x,\xi) = \sum_{l \le j} \xi_l x_{il}$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

Google Research

By the law of total probabilities:

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi) \le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x,\xi) > K_i}\right]$$

$$= \mathbb{E}\left[S_j^i(x,\xi) \cdot 1_{S_j^i(x,\xi) \le K_i}\right] + \mathbb{E}\left[K_i \cdot 1_{S_j^i(x,\xi) > K_i}\right]$$

$$= \sum_{l \le j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x,\xi) \le K_i}\right] \cdot x_{il} + K_i \cdot \mathbb{P}\left[S_j^i(x,\xi) > K_i\right]$$

$$S_j^i(x,\xi) = \sum_{l \le j} \xi_l x_{il}$$

Non-linear function because coefficients also depends on x

Google Research

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

But we can use another first-stage assignment x' too

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] = \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x',\xi)\le K_i}\right] + \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\} \cdot 1_{S_j^i(x',\xi)> K_i}\right]$$

$$\le \mathbb{E}\left[S_j^i(x,\xi) \cdot 1_{S_j^i(x',\xi)\le K_i}\right] + \mathbb{E}\left[K_i \cdot 1_{S_j^i(x',\xi)> K_i}\right]$$

$$= \sum_{l\le j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x',\xi)\le K_i}\right] \cdot x_{il} + K_i \cdot \mathbb{P}\left[S_j^i(x',\xi) > K_i\right]$$

Linear upper bound for a fixed x'

$$S_j^i(x,\xi) = \sum_{l\le j} \xi_l x_{il}$$

Key idea: How to compute $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

Google Research

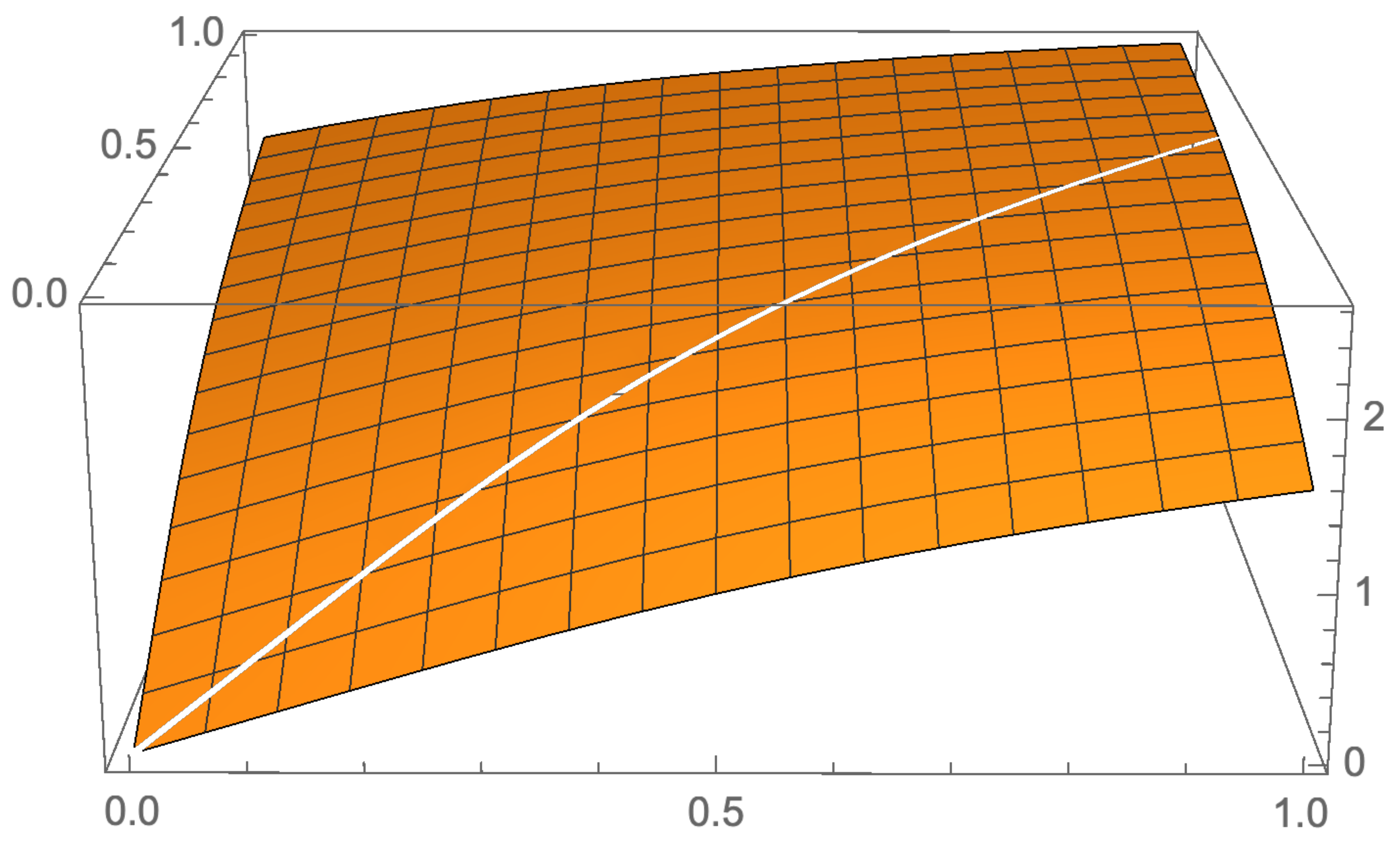
Lemma: $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$ is a concave function at x and

$$h_j^i(x,x') := \sum_{l \le j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x',\xi) \le K_i}\right] \cdot x_{il} + K_i \cdot \mathbb{P}\left[S_j^i(x',\xi) > K_i\right]$$

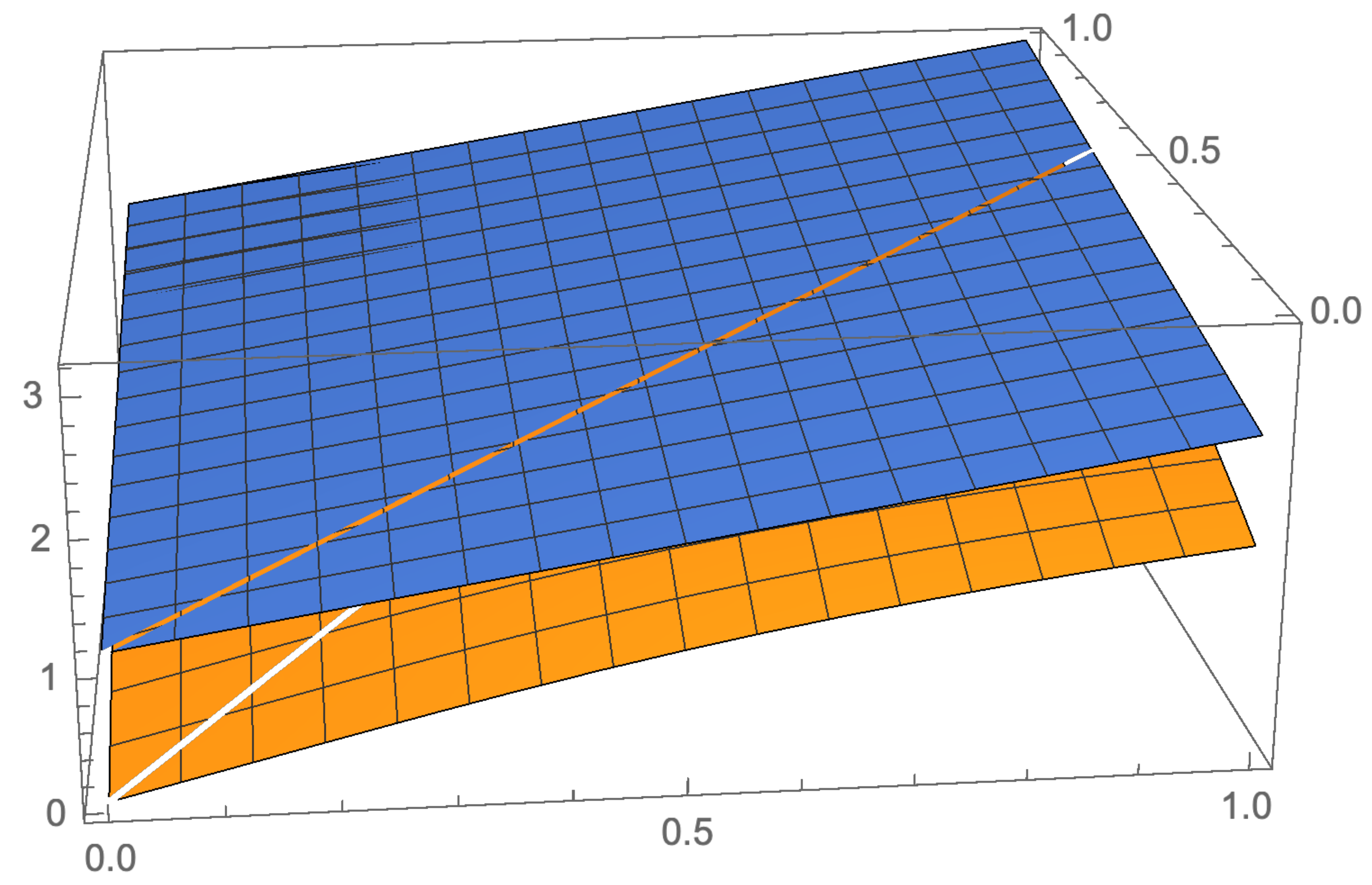
is in its subdifferential at x'

Example:

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$ with K=3



$\mathbb{E}\left[\min\{S_2(x,\xi),3\}\right]$



$h_j^i(x,x')$ at $x' = (0,1)$

Benders formulation for general distributions

We add a variable $z_{ij} \geq 0$ which correspond to the value of $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

$$\min_{x,y,z} \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{j \in J} g_{ij} \mu_j x_{ij} - \sum_{i \in I} \sum_{j \in J} (c_{i,\sigma^i(j)} - c_{i,\sigma^i(j+1)}) z_{ij}$$

$$(x,y) \in \mathcal{X} \qquad x,y \in \{0,1\}$$

It can be solve by iteratively adding Generalized Benders optimality cuts for the given incumbent solution (x', y')

$$z_{ij} \leq \sum_{l \leq j} \mathbb{E}\left[\xi_l \cdot 1_{S_j^i(x',\xi) \leq K_i y_i'}\right] \cdot x_{il} + K_i \cdot y_i \cdot \mathbb{P}\left[S_j^i(x',\xi) > K_i y_i'\right]$$

Stochastic facility location problems with outsourcing costs

1. Bender formulation for a discrete set of scenarios (for example, a sample average approximation of the demand distributions)

2. Bender formulation for general distributions.

3. **Strengthened formulation.**

4. Computational experiments

Google Research

Valid constraints on z_{ij} variables. $\left(z_{ij} \approx \mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]\right)$

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] \le \mathbb{E}\left[S_j^i(x,\xi)\right] \quad \Rightarrow \quad z_{ij} \le \sum_{l \in J : \sigma^i(l) \le \sigma^i(j)} \mathbb{E}[\xi_l] x_{il}$$

$$\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right] \le K_i \quad \Rightarrow \quad z_{ij} \le K_i y_i$$

$$S_j^i(x,\xi) = S_{j-1}^i(x,\xi) + \xi_j x_{ij} \quad \Rightarrow \quad z_{ij} \ge z_{i,j-1}$$

$$z_{ij} \le z_{i,j-1} + \mathbb{E}[\xi_j] x_{ij}$$

Submodularity of $\mathbb{E}\left[\min\{S_j^i(x,\xi), K_i\}\right]$

A set-valued function is sub modular if it has "diminishing returns".

- **Lemma**: Set-valued function $\mu(A) := \min\{S_j(1_A, \xi), K_i y_i\}$ is submodular for a given x, y, ξ.

- **Corollary**: Set-valued function $\mu'(A) := \mathbb{E}\left[\min\{S_j(1_A, \xi), K_i y_i\}\right]$ is sub-modular for a given x, y.

Google Research

Submodularity of $\mathbb{E}\left[\min\{S^i_j(x,\xi), K_i\}\right]$

Since $\mathbb{E}\left[\min\{S^i_j(x,\xi), K_i\}\right]$ is submodular, we can add submodular cuts (*)

$$z_{ij} \leq \mathbb{E}[\min\{S_j(x'), K_i y_i\}] \cdot y_i + \sum_{l:x'_{il}=0} \mathbb{E}[\xi_l 1_{S_j(x')+\xi_l \leq K_i} + (K_i - S_j(x'))1_{S_j(x') \leq K_i < S_j(x')+\xi_l}] \cdot x_{il}$$

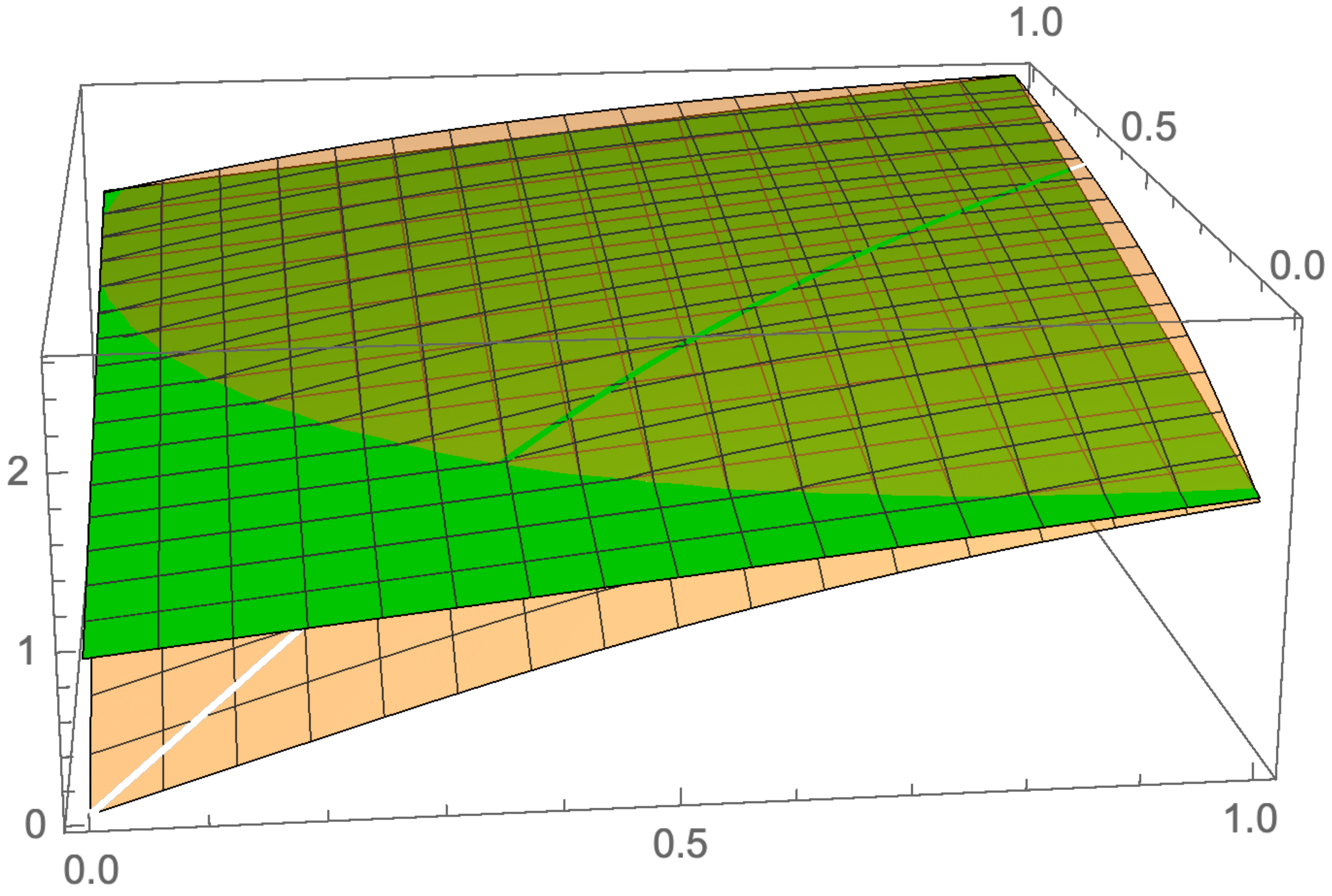
$$- \sum_{l:x'_{il}=1} \mathbb{E}[\xi_l 1_{\mathcal{S}_j \leq K_i} + (K_i - (\mathcal{S}_j - \xi_l))1_{\mathcal{S}_j - \xi_l \leq K_i < \mathcal{S}_j}] \cdot (1 - x_{il})$$

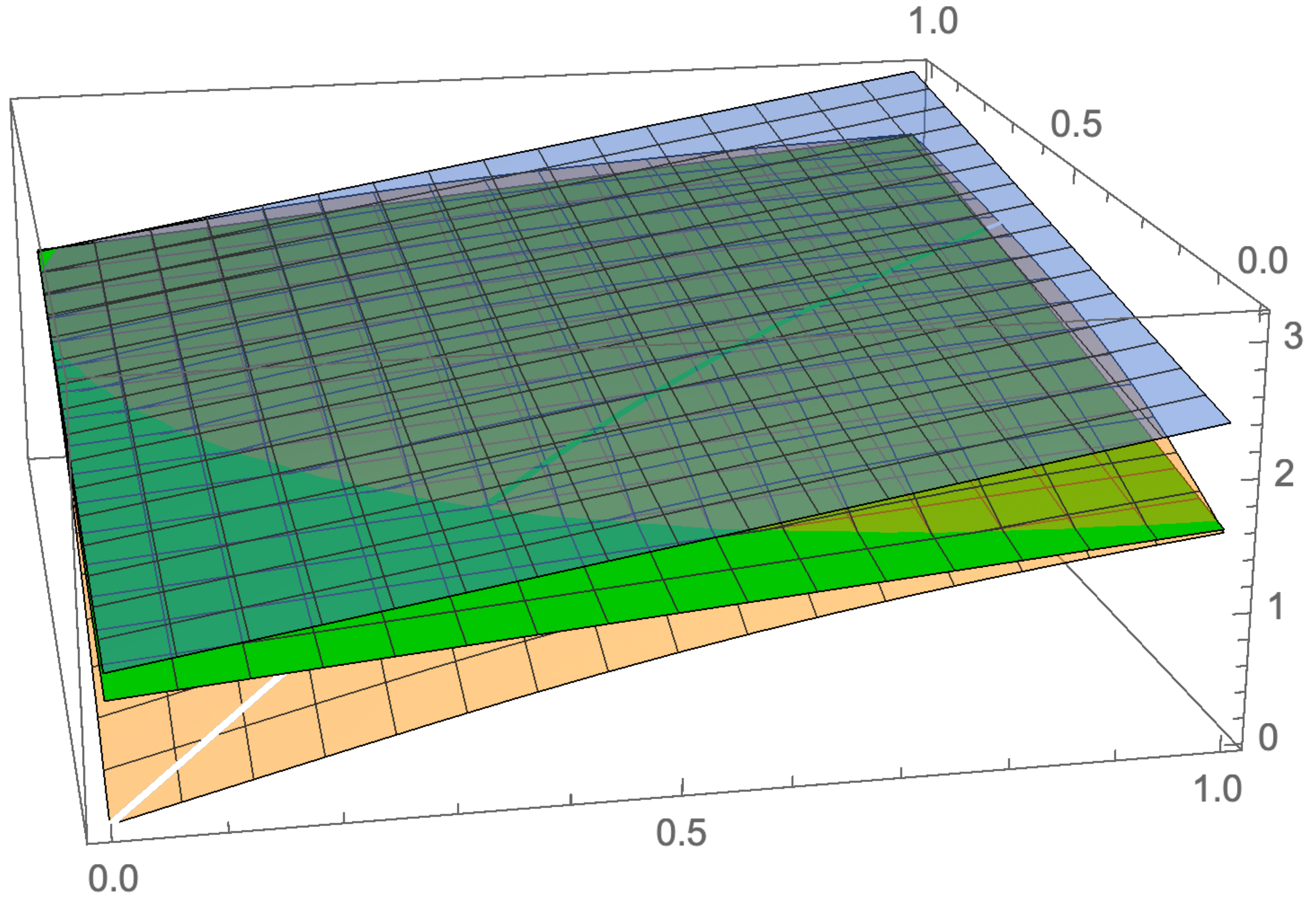
(*) see Nemhauser & Wolsey (1981), Ljubic & M. (2018)

Google Research

Example:

Two demands $\xi_1 \rightsquigarrow \exp(1/2)$ and $\xi_2 \rightsquigarrow \exp(1/3)$ with K=3





Submodular cut at $x' = (0,1)$

Piecewise linear upper bound for fractional solutions.

Both Bender and submodular cuts requires an integer solution x' to compute the coefficients to generate a cut. Can we create cuts for the relaxation of the problem?

If $\sum_{l \in J : \sigma^i(l) \leq \sigma^i(j)} x_{ij} = \kappa \in \mathbb{N}$, we can consider to sum the κ "worst" customers.

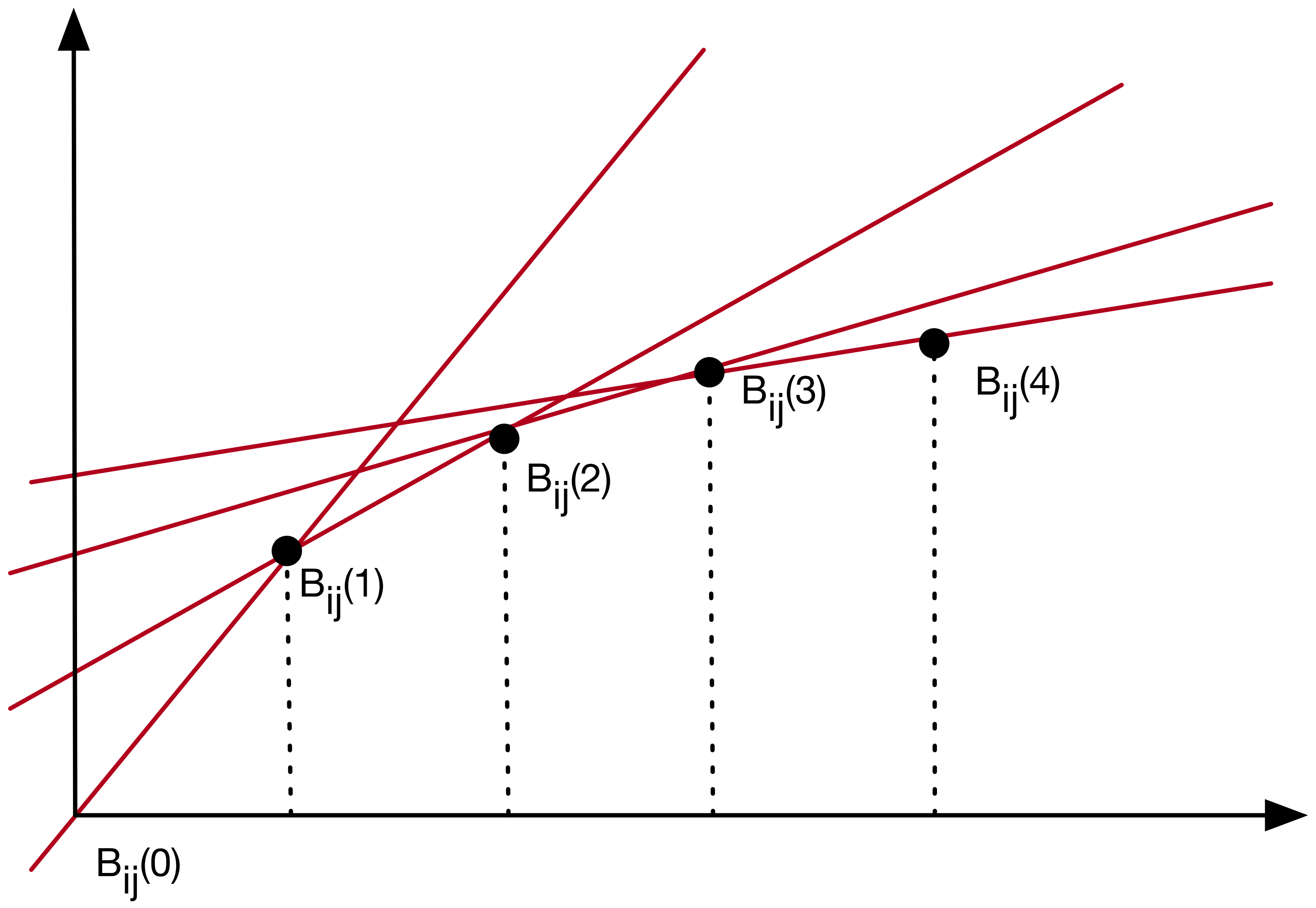
Proposition: Assume that random demands can be ordered in the usual stochastic order $\xi_{(1)} \geq_{st} \xi_{(2)} \geq_{st} \ldots \xi_{(j)}$. Then

$$\mathbb{E}\left[\min\{S_j(x), K_i y_i\}\right] \leq \mathbb{E}\left[\min\left\{\sum_{l=1}^{\kappa} \xi_{(l)}, K_i y_i\right\}\right]$$

Piecewise linear upper bound for fractional solutions.

Let $B_{ij}(\kappa) := \mathbb{E}[\min\{\mathcal{S}_\kappa, K_i y_i\}]$ the expected value considering the "worst" κ customer. We can extend this function using a piece-wise linear function, creating the valid upper bound:

$$z_{ij} \leq B_{ij}(\kappa) + (B_{ij}(\kappa+1) - B_{ij}(\kappa)) \cdot \left(\sum_{l \in J: \sigma^i(l) \leq \sigma^i(j)} x_{il} - \kappa \right)$$



- For i.i.d. demand distributions, $\mathcal{S}_\kappa$ is the sum of κ random variables. The bound is tight.
- For single-parameter distribution (e.g. exponential or Poisson) the κ worst customers are the one with higher expected demand. The bound is not tight.

Stochastic facility location problems with outsourcing costs

1. Bender formulation for a discrete set of scenarios (for example, a sample average approximation of the demand distributions)

2. Bender formulation for general distributions.

3. Strengthened formulation.

4. **Computational experiments**

Google Research

Computational experiments

Dataset for benchmarking:

- Albareda-Sambola et al (2011). 297 instances based on TSP problems, with 15 facilities and 30 customers.

- Random demands with Bernoulli and Exponential distributions with mean value 0.1, 0.5 or 0.9.

- Comparison of Benders, Submodular and PWL cuts using multi-cut (one cut for each z_{ij}) or a single-cut (aggregated cut for each facility).

- Coded in C++ using Gurobi as solver.

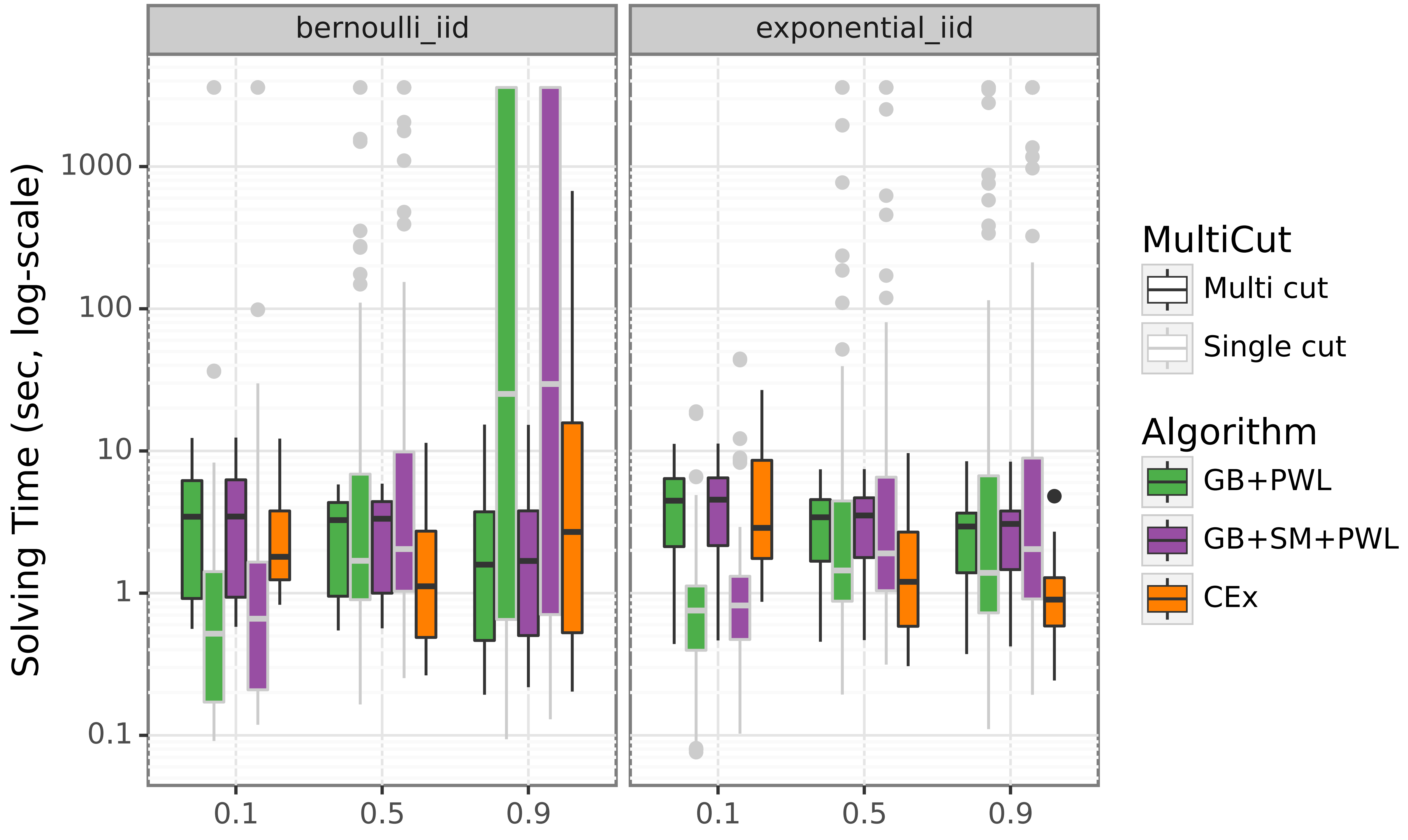
Google Research

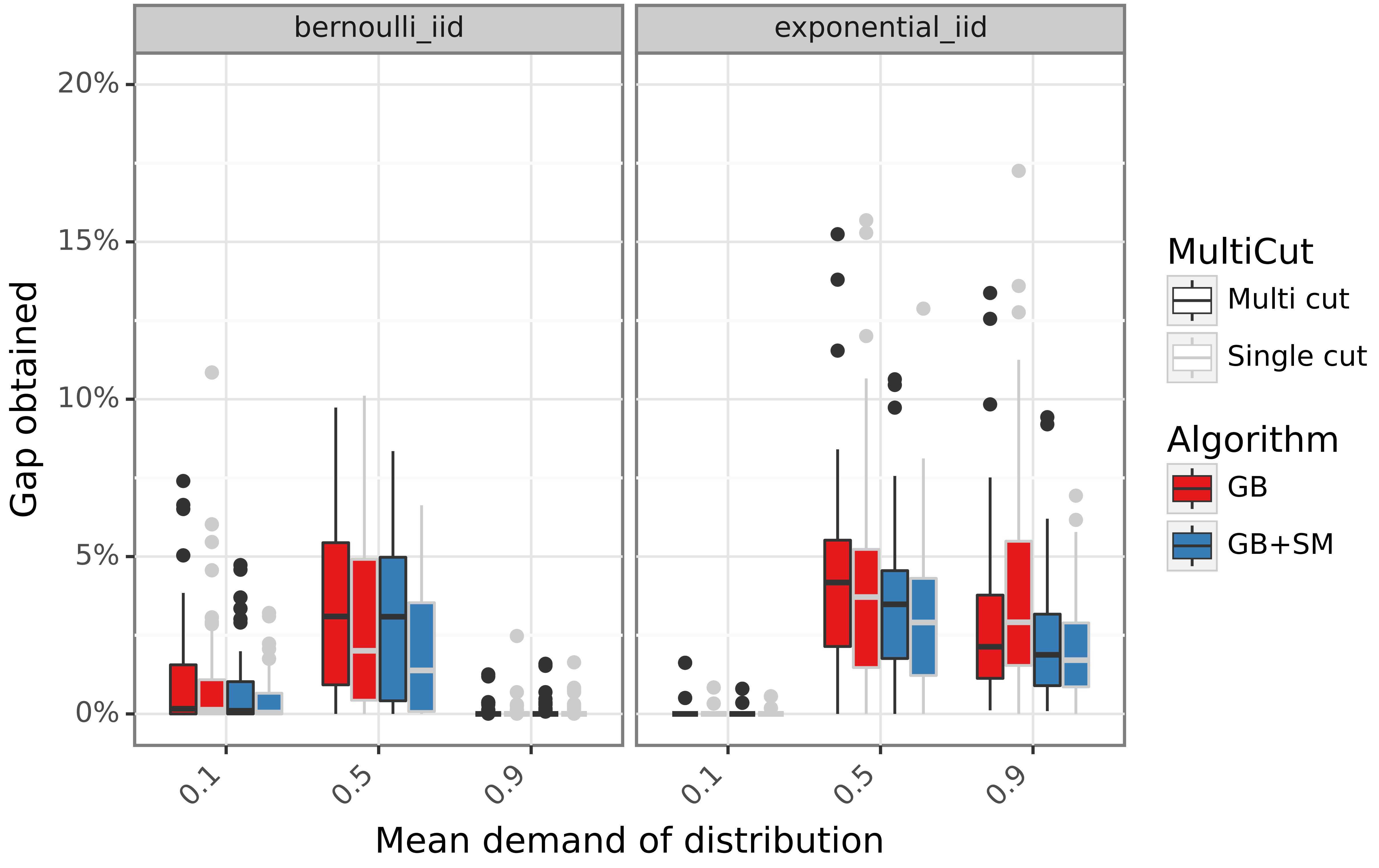
Performance profile for i.i.d. demands

- Adding PWL cuts solved all problems in a few seconds
- 50%/40% of instances are solved up to optimality. ~80% with <5% gap.

Google Research

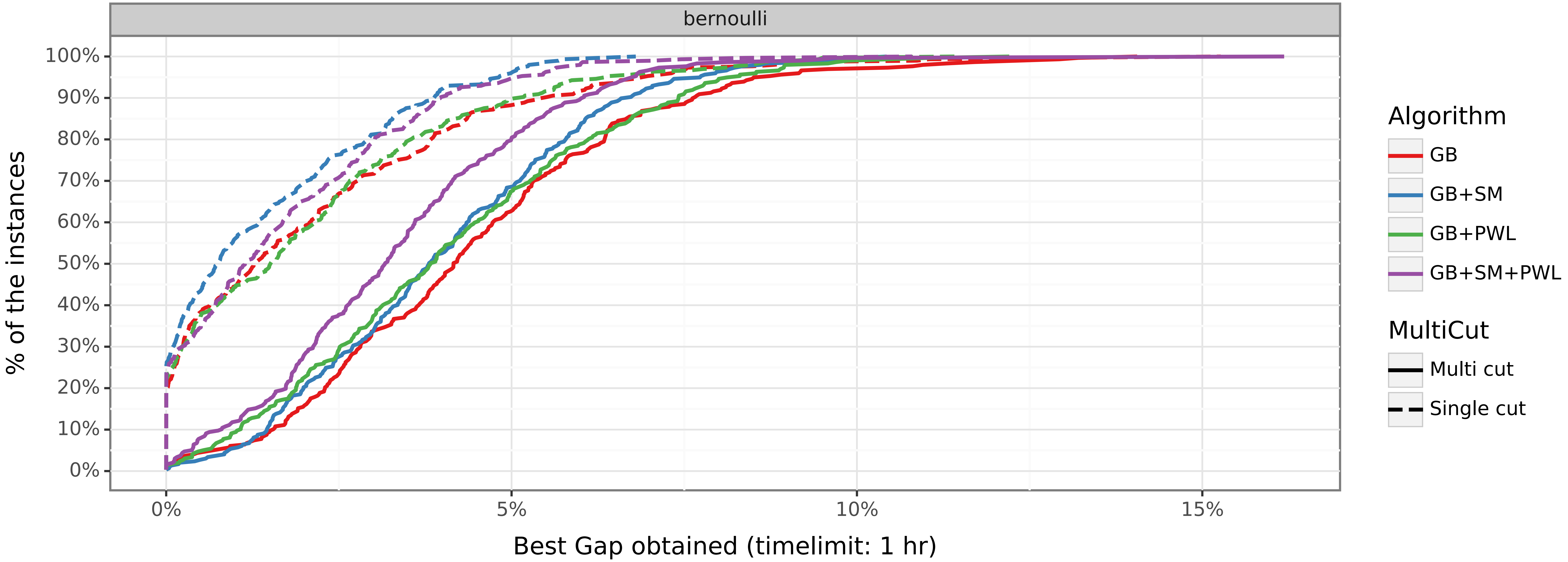
Performance profile for i.i.d. demands





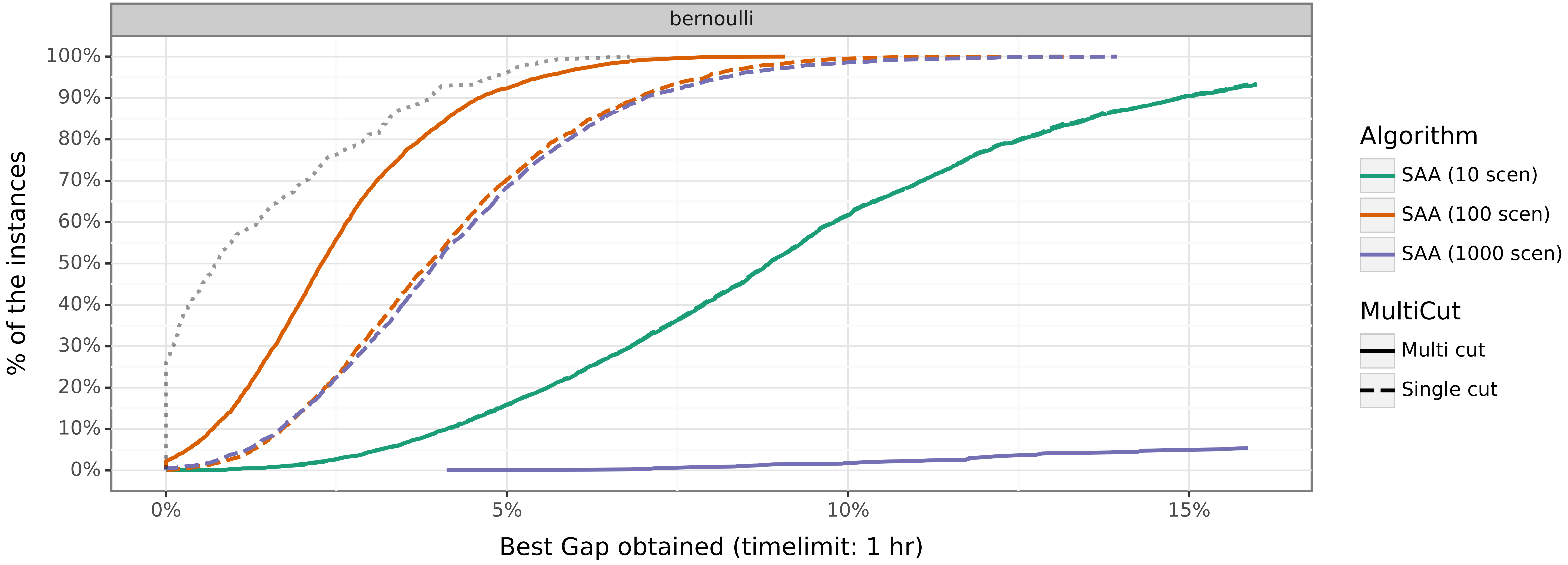
Google Research

Performance profile for non-i.i.d. demands



- Too many cuts. Aggregating cuts performs better.
- PWL cuts are not longer efficient. Submodular cuts improve the performance
- Still > 90% of instances solved with <5% gap.

Comparison with sample average approximation



- SAA with 10 scenarios is solved up to optimality, but solution quality is very bad.
- Adding more scenarios improves the quality but became harder to solve.
- Generalized Bender outperforms SAA, particularly for smaller gaps.

Conclusions

- Benders methodology for two-stage assignment problem where the second stage is a stochastic knapsack problem
  - An exact solution to the problem is achievable, precluding the necessity for scenario sampling.
  - We can exploit the structure of the subproblem: small number of optimal dual solutions where we can compute the expectations by conditioning.
  - Not an approximation! Provide true bounds for the problem.
  - Similar ideas can be extended to other problems.
  - See also Benders Adaptive Partition cuts (Ramirez-Pico & M., *Math Prog 2022*, Ramirez-Pico, Ljubic, M., *Transp Sci 2023*).