



# Integrated traffic-transit stochastic equilibrium model with park-and-ride facilities <sup>☆</sup>



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## ABSTRACT

We propose an Integrated Stochastic Equilibrium model that considers both private automobile traffic and transit networks to incorporate the interactions between these two modes in terms of travel time and generalized costs. In addition, in the general version of the model, travelers are allowed to switch from personal vehicles to mass transit at specific locations in a park-and-ride scheme. The assignment for traffic equilibrium is based on the Markovian Traffic Equilibrium model of Baillon and Cominetti (2008), whereas the equilibrium of the transit network is represented by the Stochastic Transit Equilibrium model of Cortés et al. (2013). Stochastic travel decisions are made at the node level, thereby avoiding the enumeration of routes or strategies and incorporating various perception and uncertainty issues. We propose a Method-of-Successive-Averages algorithm to calculate an Integrated Stochastic Equilibrium and conduct numerical experiments to highlight the effect of stochasticity on equilibrium flows and travel times. Our experiments show that higher stochasticity implies greater dispersion of equilibrium flows and longer expected travel times. Results on a real network with mode combination and park and ride facilities provide insights regarding the use of park and ride in terms of number and location, potential modal share of the combined mode option under different circumstances, and travel time impact due to the implementation of such park and ride facilities in a real setting.

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## 1. Introduction

Over the past two decades, many cities worldwide have grown considerably in terms of both population and land use, thereby generating new demands on transportation offerings for their inhabitants. This has motivated the interest of researchers and practitioners in modeling urban networks at different scales for various purposes. In this context, the use of urban planning models to assess investment policies for improving the welfare of people has become an important issue. At a strategic level of analysis, the so-called assignment and user equilibrium models are designed to reproduce the observed

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behavior and choices of individuals with respect to transit and traffic networks. On the one hand, transit equilibrium models seek to reproduce the boarding and alighting stops and route choices in terms of the utilized transit lines; on the other hand, traffic equilibrium models seek to reproduce the route choices in an urban road network. In the specialized literature, most articles on assignment and equilibrium models that explore pure modes, focused their analysis on the case of traffic networks, which generally rely on Wardrop's principle. Wardrop's principle states that rational users select routes that minimize their expected travel time. In turn, many existing transit equilibrium models have adopted this principle. However, there is an irremediable difference between traffic and transit modes: while choosing the route that minimizes the on-board expected travel time is sufficient in the case of traffic, in the transit dimension, the route choice is defined by the particular bus that a passenger boards within a set of different common lines that serve a bus stop that can be used to reach the destination. Subsequently, in addition to considering the on-board travel time when using the vehicle, in the transit dimension, the waiting time also plays an important role and is linked to other variables inherent to any transit system such as frequency and bus capacity.

The majority of the recent literature has focused on modeling preferences by assuming that passengers choose a route strategy for their trips. Inspired in [Chriqui and Robillard \(1975\)](#), [Spiess and Florian \(1989\)](#) defined a strategy as a set of rules that, when applied, allow a passenger to reach his/her destination. A well-formulated strategy includes the choice of the attractive lines set at a stop; furthermore, this concept assumes that users have complete knowledge of the network structure and the conditions for recognizing and using effective strategies ([Bouzaïene-Ayari et al., 2001](#)), which may appear unrealistic in situations with high traffic congestion and interaction of modes. Over the past 20 years, there has been a trend in transportation policies toward improving public transportation attractiveness by decreasing the volume of cars moving on streets and encouraging modal interchange. Hence, *park-and-ride* (P&R) facilities have emerged in specific locations of urban zones to facilitate the first leg of the trip being conducted using a personal car, followed by the second leg completing the trip through a massive and efficient mode of public transit, namely trains, buses, or subways. These trips are performed by a non-unique transportation mode known in the literature as combined modes. The incentive for users to choose these combined modes is associated with congestion on the streets, frequency and fares of transit services, and the location of parking facilities. To better reflect what occurs in large urban centers, we address the previous issues by means of combining two features: mode integration and stochasticity in travel decisions.

The goal of the present paper is to develop an Integrated Stochastic Equilibrium model that considers both traffic and transit networks to incorporate the interactions between the two pure modes in terms of travel time and generalized costs. The integrated formulation combines the Markovian Traffic Equilibrium (MTE) model developed by [Baillon and Cominetti \(2008\)](#) for the traffic network and the Stochastic Transit Equilibrium (STE) model of [Cortés et al. \(2013\)](#) for the transit network. Both models share similarities in their formulation because travel decisions are made in both cases at the node level, thereby avoiding the enumeration of routes or strategies. Moreover, both approaches include the effect of congestion at the vehicular and passenger levels in addition to stochasticity as a central feature, thereby allowing the inclusion of the various perceptions and uncertainty issues that people have regarding the features and conditions of the urban network. We emphasize that in this work *stochasticity* refers only to users perception of the level of service and not to other sources of uncertainty, such as stochasticity of travel demand and network supply. In addition, our model adds the combined mode option into the analysis, allowing users to transfer from car to transit at P&R facilities. More recently, [Liu and Meng \(2014\)](#) proposed a stochastic model on multimodal network, focused on bus-based P&R services, as well as elastic demand and congestion pricing charges. Although a probit-based stochastic transit equilibrium is assumed, which is relatively a new topic in literature, and the interaction between cars and buses in terms of travel time are largely analyzed, the framework proposed follows the common lines method adopted in [De Cea and Fernández \(1993\)](#), solving a mixed integer programming problem in order to determining the attractive line set between consecutive nodes in the transit network, and hence, computing a strategy for each transit user. Using the STE model in the integrated formulation that we propose, avoids the neither enumeration nor computation of routes or strategies.

To make this proposal applicable to real modeling conditions, we also propose an algorithm that performs the resulting stochastic equilibrium over a generic traffic and transit network, which is tested with real data of a city using, on the transit side, a logistic function for the boarding probability at bus stops and, on the traffic side, a Gumbel distribution for the error term in expected travel times. In this paper, in addition to developing the integration details, we apply the algorithm to a realistic case study corresponding to a medium-sized network of the city of Iquique, Chile.

The primary objective of the Integrated Stochastic Equilibrium model is to become an urban planning tool for transportation decision makers. The model should be calibrated in a case-by-case modality to be able to reproduce the current passenger and vehicle flows observed on the streets. Such a tool can thereby become a powerful prediction model of user behavior when relevant changes to public transportation or road infrastructure supply are implemented.

The remainder of this paper is organized as follows. First, we present a literature review of related topics. In the next section, we describe the transportation network and introduce the notations used throughout the paper. Next, in Section 3, we provide the definitions of Integrated Stochastic Equilibrium with Pure and Combined Modes. Then, in Section 4, the solution algorithms are presented. In Section 5, we present the results of various experiments to demonstrate the potential and consistency of the stochastic formulation under different scenarios. In Section 6 we report the implementation of the model in the real network of Iquique. The paper closes with a summary, conclusions and ideas for further work.

### 1.1. Literature review

The literature on transit equilibrium and passenger assignment goes back to [Chriqui and Robillard \(1975\)](#), [Nguyen and Pallotino \(1988\)](#) and [Spiess and Florian \(1989\)](#) who rely on the concepts of strategies and hyperpaths to develop uncongested transit equilibrium models. [Cominetti and Correa \(2001\)](#) developed a transit equilibrium model based on hyperpaths that explicitly includes the effects of congestion through a queuing model at bus stops. [Cepeda et al. \(2006\)](#) extended the previous model and obtained a new characterization of equilibrium. These models consider that passengers choose the optimal strategy or hyperpath that minimizes the generalized travel costs, thus leading to a deterministic approach. In the context of transit equilibrium, crowding effects have been included and modeled by [Rouwendal and Verhoef \(2004\)](#), [Tian et al. \(2007\)](#) and, more recently, [de Palma et al. \(2015\)](#).<sup>1</sup> Stochastic assignment with congestion is introduced by [Nguyen et al. \(1998\)](#) who proposed a nested logit model considering every set of competitive transit lines as a hyperpath. [Xu et al. \(2012\)](#) proposed a stochastic user equilibrium model for a scheduled-based transit network with capacity constraints. Later on, [Cortés et al. \(2013\)](#) extended [Cepeda et al. \(2006\)](#) by proposing a stochastic approach that considers both congestion at bus stops and the stochastic behavior of passengers during the boarding process.

At this point of the review, it is worth mentioning other innovative approaches related to transit assignment recently published. [Ma and Fukuda \(2015\)](#) used the hyperpath concept as part of a network generalized extreme-value model (denoted N-GEV) for route choice under uncertainties, incorporating both random and deterministic aspects caused by potential travel delay elements (in-vehicle and waiting times variability) or network disruption risks (traffic incidents). [Moyo Oliveros and Nagel \(2016\)](#) studied how behavioral microscopic rules behind an agent-based simulation scheme should be modified for obtaining results from the assignment closer to observed passenger flow counts. The authors show promising results from the calibration and simulation processes, applied on a real network coded for the city of Berlin; it must be noticed that using this approach requires a highly detailed and reliable user activity data apart from travel data itself, which in many cases is difficult to collect. [Sun et al. \(2015\)](#) presented a Bayesian inference approach that makes use of smartcard data in a very intensive way. The model estimates network attributes (such as the distribution of link travel time variability) as well as passenger route choice preferences using only observations of travel time in an isolated subway network. However, the high number of unknown variables representing the attributes that appear in more complex networks could make the calculation processes computationally difficult.

In the field of vehicle assignment on traffic networks, research has been focused on the development of stochastic models that consider the variability among users in perceiving the travel costs. If error terms in routing costs are modeled as independent Gumbel random variables, the well-known logit model for route choice is obtained.<sup>2</sup> [Daganzo and Sheffi \(1977\)](#) proposed an alternative model based on a probit formulation for the stochastic assignments.<sup>3</sup> To partially address the issue of overlapping routes [Bekhor and Prashker \(2001\)](#) proposed a generalized nested logit model, while [Kitthamkesorn et al. \(2013\)](#) formulated a nested logit model for modal split and a cross-nested logit model to account for route overlapping, capturing in a better way the different degrees of overlapping among available routes. [Baillon and Cominetti \(2008\)](#) propose a stochastic user equilibrium model based on a discrete choice at each intermediate node under a sequential arc selection process, rather than basing the decision on the entire route, avoiding enumeration of paths, which can become computationally impractical for large networks. The reader can extend the search of other traffic equilibrium models considering uncertainty.<sup>4</sup>

The literature of P&R schemes have been oriented mostly to find optimal location of parking facilities ([Wang et al., 2004](#); [Holguin-Veras et al., 2012](#); [Farhan and Murray, 2008](#)). Few works have studied equilibrium conditions and multimodal choice models in this context. Among these, we can mention [Fernández et al. \(1994\)](#), that formulate a user equilibrium including combined modes and modeled the choice of transfer nodes through a nested logit model assuming symmetric cost functions; [García and Marín \(2005\)](#), who extended the previous model including asymmetric cost using a 2-step nested logit formulation: mode choice at the first step and transfer node at the second (route choice is deterministic); [Li et al. \(2007\)](#), who developed an equilibrium formulation that can be used to model P&R services in a multimodal network with elastic demand showing from numerical results that P&R schemes could bring a positive, neutral or even negative welfare increment. This last work focuses in deterministic networks, and travel time variability or other stochastic effects are ignored. [Liu et al. \(2009\)](#) performed a bi-modal analysis in a competitive railway and highway system. [Wang et al. \(2014\)](#) proposed an optimal parking fare scheme for P&R, which is formulated as a bilevel program: in the upper-level, the problem refers to find the optimal parking fare for improving network performance, and in the lower-level, the problem is to evaluate network performance in equilibrium.

<sup>1</sup> [Rouwendal and Verhoef \(2004\)](#) consider crowding as an increasing function of occupancy, while [Tian et al. \(2007\)](#) analyze crowding as a function of number of passengers along with seat allocation. [de Palma et al. \(2015\)](#) incorporated crowding in public transport, highlighting its implications in pricing, seating capacity and optimal scheduling in the context of user equilibrium and system optimal problems.

<sup>2</sup> See [Fisk \(1980\)](#), [Yang et al. \(2001\)](#), [Meng et al. \(2004\)](#), [Cascetta et al. \(1996\)](#) and [Zhou et al. \(2012\)](#) for C-logit based stochastic user equilibrium models.

<sup>3</sup> Alternative formulations based on probit models have been proposed by [Connors et al. \(2007\)](#), [Uchida et al. \(2007\)](#), and [Meng et al. \(2012\)](#) among others.

<sup>4</sup> Utility-based models ([Yin and Ieda, 2001](#); [Chen et al., 2002](#); [Di et al., 2008](#)), game theory based approaches ([Bell, 2000](#); [Bell and Cassir, 2002](#); [Szeto et al., 2006](#)), prospect theory-based approaches ([Connors and Sumalee, 2009](#); [Sumalee et al., 2009](#); [Xu et al., 2011](#); [Xu et al., 2011](#)), reliability-based models ([Chen and Zhou, 2010](#); [Chen and Zhou, 2011](#)).

## 2. Preliminaries

We are interested in studying the movement of people from different origins to different destinations in an urban transportation network over a predefined period of time. In our approach, the network consists of transit and private cars only.

To represent this network, consider a directed graph  $G = (N, A)$ , and denote by  $i_a$  and  $j_a$  the tail and head nodes of an arc  $a \in A$ . Let  $A_i^+ = \{a \in A : i_a = i\}$  and  $A_i^- = \{a \in A : j_a = i\}$  be the sets of outgoing and incoming arcs from/to node  $i \in N$ , respectively. Note that the set  $A$  consists of all types of arcs, including *on-board*, *alighting*, *boarding* (for transit), *road* (for traffic) and *walking* arcs. Let  $d \in D \subseteq N$  be the subset of destination nodes within the network. For each  $d \in D$  and every node  $i \neq d$ , a fixed demand  $g_{id} \geq 0$  is given. An efficient way to represent flows on arcs is to specify arc-destination flows. The set  $\mathcal{V} := \mathbb{R}_+^{|A| \times |D|}$  denotes the space of arc-destination flow vectors  $v$  with nonnegative entries  $v_{ad} \geq 0$ , whereas  $\mathcal{V}_0$  is the set of feasible flows  $v \in \mathcal{V}$  such that  $v_{ad} = 0$  for all  $a \in A_d^+$  (i.e., no flow with destination  $d$  leaves from  $d$ ) and satisfying the flow conservation constraints:

$$g_{id} + \sum_{a \in A_i^-} v_{ad} = \sum_{a \in A_i^+} v_{ad} \quad \forall i \neq d. \tag{1}$$

Let  $v_a = \sum_{d \in D} v_{ad}$  be the total flow on arc  $a$ .

The nodes  $i, d$  for which  $g_{id} > 0$  are denoted *centroids* and represent urban areas where demand is generated and attracted. These nodes are connected to two separated subnetworks, namely, a transit and traffic network, through *walking* or *connector* arcs, respectively.

The transit network is composed of bus stops and bus lines that serve the bus stops. Each bus stop is represented by several nodes: one node is associated with the platform where people wait (*stop nodes*), and the remaining nodes are associated with a line that stops at that bus stop (*line nodes*). For each line  $l$  that serves a stop  $s$ , the line node  $h_s^l$  is connected to the stop node  $s$  by one *boarding* arc and one *alighting* arc, both of which must be properly specified (see Fig. 1). We denote  $N_S$  and  $N_L$  as the sets of stop and line nodes of the transit network, respectively.

Each arc  $a \in A_i^+$  is associated with a line that stops at node  $i$  and represents either the journey between two consecutive bus stops or the boarding/alighting processes at a given bus stop. Each arc  $a$  in the transit network is characterized by a continuous travel time function  $t_a : \mathcal{V} \rightarrow [0, \bar{t}_a[$ , where  $\bar{t}_a$  is a finite upper bound, and the effective frequency function  $f_a : \mathcal{V} \rightarrow [0, +\infty]$ . The effective frequency is either equal to  $+\infty$  or everywhere finite, in which case, for each  $d \in D$ , we assume that  $f_a \rightarrow 0$  when  $v_a \rightarrow \bar{v}_a$ , where  $\bar{v}_a$  is the capacity of the line represented by arc  $a$ , with  $f_a(v)$  strictly decreasing with respect to  $v_a$  when strictly positive. These functions reflect the congestion of transit lines. In particular, when a line is completely congested, the observed frequency of that line by a passenger waiting at that stop is zero. The travel time and frequency functions of an arc may depend on the flow on other arcs in the network  $G$ . For instance, for a boarding arc, the frequency function may depend on the flow of the arriving line segment arc; and the travel time function of an on-board arc may depend on the flow of an arc representing a street on the traffic network where that bus circulates (when the flow of transit is not separated from traffic in the urban network).

The structure of the transit network is illustrated in Fig. 1. Centroid  $d$  is connected to stop node  $s$  through walking arcs that have frequencies  $f_a = \infty$  because these arcs are always available once the passenger decides to go to a certain stop and have no waiting time associated with them. The travel time on these arcs is equal to the walking time between the zone and the stop. For boarding arcs  $(s, h_s^l)$ , the frequency is equal to the effective frequency  $f_a$  that the line has on that stop, whereas the travel time  $t_a$  in these arcs is assumed to be negligible and equal to zero, although the model is sufficiently flexible to set these variables to a value greater than zero, including passenger transfer time at stops. For alighting arcs  $(h_s^l, s)$ , the travel times are also set as  $t_a = 0$ , but the frequencies are fixed to  $f_a = \infty$  because a passenger does not have to wait to alight the bus once it has stopped. Finally, we have arcs  $(h_s^l, h_{s'}^l)$  that represent the on-board travel on line  $l$  between two consecutive bus stops  $s$  and  $s'$ , where  $f_a = \infty$  because a passenger already boarded the bus.

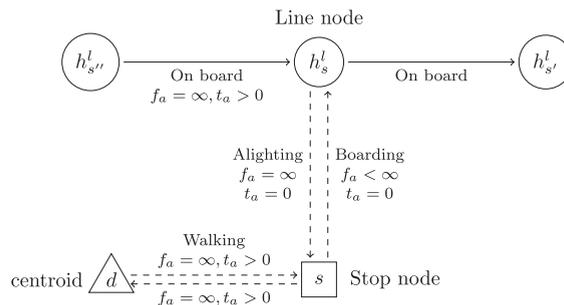


Fig. 1. Stop representation on the transit network.

The traffic network is composed of nodes that represent intersections and arcs between nodes that represent road segments. Each arc is characterized by a strictly increasing continuous travel time function of the flow on the arc,  $s_a : \mathbb{R} \rightarrow (0, \infty)$ , which models vehicle congestion. Centroids are connected to one or more nodes in the traffic network through connector arcs with travel times equal to zero.

### 3. Equilibrium

In this section we present our equilibrium concept. To define the Integrated Stochastic Equilibrium we first need to remind the definitions of STE (Cortés et al., 2013) and MTE (Cominetti and Correa, 2001).

#### 3.1. Stochastic transit equilibrium

The concept of stochastic transit equilibrium was developed by Cortés et al. (2013), who extended the deterministic formulation of Cominetti and Correa (2001) and Cepeda et al. (2006) based on minimum hyperpath choice. This model reflects the perception of passengers in terms of the level of service of a specific line, including factors such as traffic conditions of the transit network and reliability of the line, among others. The model includes stochasticity through a probability distribution associated with boarding a bus belonging to a specific line, which can be characterized by the observed frequency at a given stop along with the expected travel time to the next stop. This formulation generates a stochastic common lines problem, in which each line has a probability of being chosen by a passenger, even if the quality of service is poor. In addition, the formulation incorporates capacity constraints at stops. A significant difference between this formulation and similar deterministic approaches in the literature is that it is no longer necessary to enlist all the feasible strategies. This is possible because the expected travel time values can be analytically computed for a given destination together with the equilibrium flows on each line, thereby simultaneously solving a set of common lines problems that are interrelated by flow conservation constraints at each node. Moreover, Cortés et al. (2013) proposed an algorithm to find the stochastic equilibrium.

Consider a passenger traveling to destination  $d$  that reaches an intermediate node  $i$  on his or her trip, as shown in Fig. 2.

To exit from  $i$ , the passenger can choose one of the arcs  $a \in A_i^+$  to reach the next node  $j_a$ . Let  $\tau_{id}$  be the expected total travel time from node  $i$  to destination  $d$ . In the common lines problem, the passenger compares the times  $t_a(v) + \tau_{j_a d}$  when choosing which arcs to follow. In the stochastic common lines problem, each passenger has probability  $p_a^d$  of wishing to board a bus of line  $a$  to reach destination  $d$ , given that a bus of line  $a$  is at the bus stop<sup>5</sup>:

$$p_a^d = \mathbb{P}(\text{boarding a bus to reach destination } d | \text{bus of line } a \text{ is at the stop}).$$

A passenger that wishes to travel from  $i$  to  $d$  compares the expected travel time of boarding the current bus,  $t_a(v) + \tau_{j_a d}$ , with the expected travel time of not boarding and waiting,  $\tau_{id}$ . This probability is given by a stochastic model and depends on the expected travel time. We will assume that  $p_a^d$  is a strictly decreasing continuous function of the difference between the expected travel time on the current bus  $t_a(v) + \tau_{j_a d}$  and the expected travel time of waiting  $\tau_{id}$ ,  $\varphi_a : \mathbb{R} \rightarrow ]0, 1[$ :

$$p_a^d \equiv \varphi_a(t_a(v) + \tau_{j_a d} - \tau_{id}). \quad (2)$$

$\varphi_a$  is such that  $\varphi_a(t) \rightarrow 0$  when  $t \rightarrow +\infty$  and  $\varphi_a(t) \rightarrow 1$  when  $t \rightarrow -\infty$ . These functions are introduced merely as tools that are used to model the stochastic decision of boarding and are to be determined by the modeler.

Cortés et al. (2013) assumed that the arrival process is completely renewed each time a bus arrives at a certain stop and that the arrival of buses follows an exponential distribution. These assumptions allow us to calculate the expected travel time and flow assignment (respectively) as in Eqs. (4) and (5) below.

The stochastic transit equilibrium is formulated as a set of simultaneous stochastic common lines problems (one for each  $id$  pair) that are coupled by flow conservation constraints (Cortés et al., 2013). We define for each  $v \in \mathcal{V}$  the flow entering node  $i$  with destination  $d$  by

$$x_{id}(v) := g_{id} + \sum_{a \in A_i^+} v_{ad}. \quad (3)$$

**Definition 1.** Given a transit network  $G = (N, A)$ , a pair of feasible flow vector and expected travel times  $(v^*, \tau^*) \in \mathcal{V}_0 \times \mathbb{R}_+^{N \times |D|}$  is a Stochastic Transit Equilibrium if for all  $d \in D$  and  $i \in N$ , with  $i \neq d$ , we have

$$\tau_{id}^* = \frac{1 + \sum_{a \in A_i^+} p_a^d f_a(v^*) (t_a(v^*) + \tau_{j_a d}^*)}{\sum_{a \in A_i^+} f_a(v^*) p_a^d}; \quad (4)$$

<sup>5</sup> Keep in mind that arc  $a$  and destination  $d$  are not necessarily directly connected.

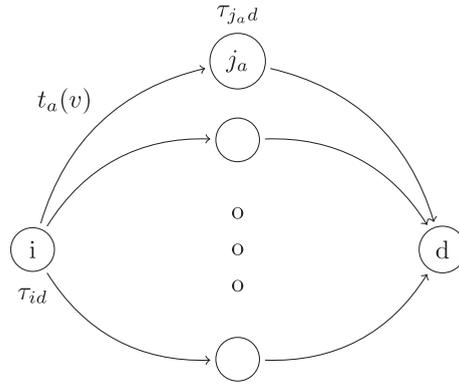


Fig. 2. Common lines problem on a general transit network.

$$v_{ad}^* = x_{id}(v^*) \frac{f_a(v^*)p_a^d}{\sum_{a \in A_i^+} f_a(v^*)p_a^d}, \quad \forall a \in A_i^+; \tag{5}$$

$$p_a^d = \varphi_a(t_a(v^*) + \tau_{j_a d}^* - \tau_{id}^*), \quad \forall a \in A_i^+. \tag{6}$$

### 3.2. Markovian Traffic Equilibrium

The basis of the Markovian equilibrium model of traffic networks in Baillon and Cominetti (2008) is that car users travel to their destinations through a sequential arc selection process based on a discrete choice model at every intermediate node  $i$  that they reach during their trip. This process is governed by an embedded Markov chain, and thus, the authors called the model Markovian Traffic Equilibrium (MTE). They proved that this formulation leads to a strictly convex minimization problem that avoids path enumeration, and they also proposed computational methods that are effective even for large networks.

Let the random variable  $\tilde{s}_a = s_a + v_a$  be the travel time on arc  $a \in A_i^+$ , where  $s_a$  is the deterministic travel time on that arc, and let  $v_a$  be an error term that represents variability between drivers' perceptions. It is assumed that these terms have a continuous distribution, where  $\mathbb{E}(v_a) = 0$ . Let  $R_{id}$  be the set of all available paths from  $i$  to  $d$ . Then, the optimal travel time from  $i$  to  $d$  is given by

$$\tilde{\tau}_{id} = \min_{r \in R_{id}} \left\{ \sum_{a \in r} \tilde{s}_a \right\}.$$

Given a destination  $d$ , a driver arrives at node  $i$  and compares the travel time or generalized costs using each of the outgoing arcs of node  $i$ . Let  $\tilde{z}_{ad}$  be the stochastic time or cost to destination  $d$  using arc  $a$ :

$$\tilde{z}_{ad} = \tilde{s}_a + \tilde{\tau}_{j_a d} = z_{ad} + \epsilon_{ad} \tag{7}$$

where  $z_{ad} = \mathbb{E}(\tilde{z}_{ad})$  and  $\mathbb{E}(\epsilon_{ad}) = 0$ . The driver selects the arc that has the shortest time between the set  $a \in A_i^+$  according to their perception. This process is repeated at each intermediate node during the trip. Then, for each destination  $d \in D$ , there is an underlying Markov chain in the network, where for a node  $i \neq d$ , the transition probabilities are given by

$$q_a^d = \begin{cases} \mathbb{P}(\tilde{z}_{ad} \leq \tilde{z}_{a'd}, \forall a' \in A_i^+) & \text{if } a \in A_i^+ \\ 0 & \text{if not} \end{cases}$$

while the destination node is an absorbing state of the chain, i.e.,  $q_{(d,d)}^d = 1$ .

Baillon and Cominetti (2008) show that, for each pair  $id$ , the probability of using arc  $a$  from node  $i$  to reach destination  $d$  may be expressed using the expected travel time function  $\psi_{id} : \mathbb{R}^{|A_i^+|} \rightarrow \mathbb{R}$  defined by:

$$\psi_{id}(z_d) \equiv \mathbb{E} \left( \min_{a \in A_i^+} \{z_{ad} + \epsilon_{ad}\} \right) \tag{8}$$

where  $z_d$  represents the vector that contains all of the values  $z_{ad}$  of the network for a given destination  $d$ . Indeed, these transition probabilities take the form

$$\mathbb{P}(\tilde{z}_{ad} \leq \tilde{z}_{a'd} \forall a' \in A_i^+) = \frac{\partial \psi_{id}}{\partial z_{ad}}(z_d). \tag{9}$$

The functions  $\psi_{id}$ , which are component-wise non-decreasing, concave and smooth, are determined by the random vector  $\epsilon_d$  and, in turn, by the variables  $v_a$ . The functions that belong to this class, denoted by  $\mathcal{E}$ , and where  $\psi_{dd} \equiv 0$ , permit an analytical characterization, as shown in [Baillon and Cominetti \(2008\)](#).

With these elements, it is possible to describe Bellman's dynamic programming equations as  $\tilde{\tau}_{id} = \min_{a \in A_i^+} \tilde{z}_{ad}$ . Taking the expectation of both sides, we have  $\tau_{id} = \psi_{id}(z_d)$ . Because Eq. (7) gives  $z_{ad} = s_a + \tau_{j_{ad}}$ , the Bellman equations can be expressed in terms of the  $\tau_{id}$  variables as follows:

$$\tau_{id} = \psi_{id} \left( \left( s_a + \tau_{j_{ad}} \right)_{a \in A_i^+} \right). \quad (10)$$

Furthermore, analogous to the STE, we have the network load and flow conservation constraints at each node as in Eqs. (5) and (3). Hence, the flow distribution system can be written as

$$\begin{aligned} x_{id} &= g_{id} + \sum_{a \in A_i^+} v_{ad} \\ v_{ad} &= x_{id} \cdot q_a^d \quad \forall i \neq d \quad \text{and} \quad \forall a \in A_i^+. \end{aligned} \quad (11)$$

Given a family of functions  $\psi_{id} \in \mathcal{E}$  (one for each  $id \in N \times D$  pair) with  $\psi_{dd} \equiv 0$ , the MTE is formalized as follows:

**Definition 2.** Given a traffic network  $G = (N, A)$ , a vector  $v^* \in \mathbb{R}^{|A|}$  is a Markovian Traffic Equilibrium if and only if  $v_a^* = \sum_{d \in D} v_{ad}^*$ , where the components  $v_{ad}^*$  satisfy the flow distribution system (11) with  $\tau_{id}$  solving (10) for  $s_a = s_a(v_a^*)$ .

In tests involving large networks, [Baillon and Cominetti \(2008\)](#) used a BPR-type formulation for travel time functions  $s_a(v_a)$  and a logit formulation family with parameters  $\beta_{id}$  for each expected travel time function  $\psi_{id}(\cdot)$ . With this, expressions (10) and (9) become

$$\tau_{id} = -\frac{1}{\beta_{id}} \ln \left( \sum_{a \in A_i^+} e^{-\beta_{id} (s_a + \tau_{j_{ad}})} \right), \quad (12)$$

$$q_a^d = \frac{e^{-\beta_{id} (s_a + \tau_{j_{ad}})}}{\sum_{a' \in A_i^+} e^{-\beta_{id} (s_{a'} + \tau_{j_{a'd}})}}, \quad (13)$$

respectively.

### 3.3. Integrated Stochastic Equilibrium

#### 3.3.1. Integrated Stochastic Equilibrium with pure mode choice

In this section, we develop a joint equilibrium model that couples the STE and MTE models. Both formulations interact at the demand level associated with each origin-destination pair and the corresponding travel times because traffic flow may affect transit travel times, as both modes may use the same road infrastructure. Conceptually, the Integrated Stochastic Equilibrium model relies on the following premise: certain users of the network perform a modal choice at their origin nodes prior to starting their trips. These users, who have access to both modes, decide whether to use a car or the bus, thereby remaining on the traffic network or being transferred to the transit network. In turn, users who have only the bus as an available option are *captive users* of the bus and move only through the transit network.

Let us denote  $U_{id}^m$  as the generalized cost of traveling from origin  $i$  to destination  $d$  by mode  $m$ , with  $m \in \{B, C\}$  representing *bus* and *car*, respectively. The modal choice for users with car and bus availability is established using a logit function with dispersion parameter  $\beta_M$ , that considers the generalized costs of traveling in both networks. Let  $G_{id}^B$  denote the number of users who can only use the bus to travel between the OD pair  $id$ ; and let  $G_{id}^C$  denote the number of users that can freely choose between the car and the bus to perform the trip on the OD pair. Then, given the values of the generalized costs, the input demand on both transit and traffic networks are

$$g_{id}^B = G_{id}^B + G_{id}^C \cdot \frac{e^{\beta_M U_{id}^B}}{e^{\beta_M U_{id}^B} + e^{\beta_M U_{id}^C}} \quad (14)$$

$$g_{id}^C = G_{id}^C - G_{id}^C \cdot \frac{e^{\beta_M U_{id}^B}}{e^{\beta_M U_{id}^B} + e^{\beta_M U_{id}^C}}, \quad (15)$$

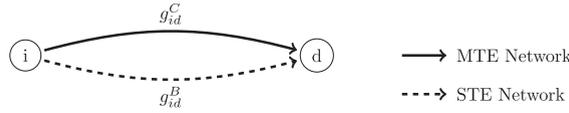


Fig. 3. Modal split on traffic and transit networks with pure modes.

respectively. Note that expression  $G_{id}^C \cdot \frac{e^{\beta_M U_{id}^B}}{e^{\beta_M U_{id}^B} + e^{\beta_M U_{id}^C}}$  represents the total number of users who observe travel costs in both modes but prefer to use the bus, making a transference from the traffic to the transit network. Graphically, the modal split process is shown in Fig. 3.

In our integrated model, the generalized cost of traveling from  $i$  to  $d$  by mode  $m$ , namely,  $U_{id}^m$ , is given by a generalized cost or utility function  $u^m : \mathbb{R}_+ \rightarrow \mathbb{R}$ .  $u^m(t)$  is the generalized cost of using mode  $m$  when the travel time on that mode is  $t$ . This utility function may include the monetary costs of using each mode – fares, fuel prices, etc. – and other issues involving modal choice such as comfort and modal attractiveness.

With these elements, we are now able to formalize the definition of integrated equilibrium with pure modes.

**Definition 3.** Given an urban transportation network  $G = (N, A)$  containing transit and traffic, a tuple  $(U^*, g^*, v^*, \tau^*)$  is an *Integrated Stochastic Equilibrium* if flows and expected travel times in the traffic network  $(v^C, \tau^C)$  are an MTE with demand  $g^C$ , flows and expected total times in the transit network  $(v^B, \tau^B)$  are an STE with demand  $g^B$  given  $v^C$ , and  $(g^*, U^*)$  satisfy Eqs. (14) and (15) with  $U_{id}^m = u^m(\tau_{id}^m)$  for all  $i \in N$  and  $d \in D$ .

### 3.3.2. Integrated Stochastic Equilibrium with combined mode choice

A natural extension of the previous model is the addition of the combined option of traveling through both networks during the realization of an OD trip. In our proposal, we consider that a traveler that can use both a car and a bus may choose to drive during the first part of the trip, park the car at an intermediate transfer station, and, from there, complete the trip by public transportation toward the final destination. This modality of travel is known as *park-and-ride*, which is a system that successfully operates in many cities worldwide (e.g., Boston, New York, Oxford, Montreal, Norwich, and Bristol). In general, the transfer stations are located in the suburbs of the city, which are places that should have good connectivity to the public transportation network and adequate facilities for parking, thus improving the use of transit systems within the city and consequently reducing the number of private cars within congested urban zones.

In this case, we decide to use a hierarchical logit formulation to represent the modal choice process with combined modes (García and Marín, 2005). There are two decision levels. First, at the upper level, the decision is performed by choosing the mode, which can be either car only, bus only, or car and bus as a combined mode. If the traveler chooses the combined option, then there is a (lower) second-level decision, in which the user must choose where to transfer between their car and the bus among a predefined set of options. The reason for having two decision levels (and therefore, not simply analyzing at the same level all possible options between modes and transfer stations) is because there is a strong correlation between the combined mode options; thus, the assumption underlying the independent alternatives behind the multinomial logit model does not apply in this case.

Let us denote mode  $m \in \{B, C, P\}$  for bus, car and combined mode, respectively, and let  $K_P \subset D$  be the set of transfer nodes within the transportation network, which allow people to transfer between cars and buses. Then, we denote by  $g_{id,k}^P$  the total number of users who travel between origin  $i$  and destination  $d$  using the combined mode  $P$  and choosing transfer node  $k \in K_P$ . Graphically, the demand split for those travelers with both pure modes available is shown in Fig. 4, where each level represents a decision that must be made by the traveler.

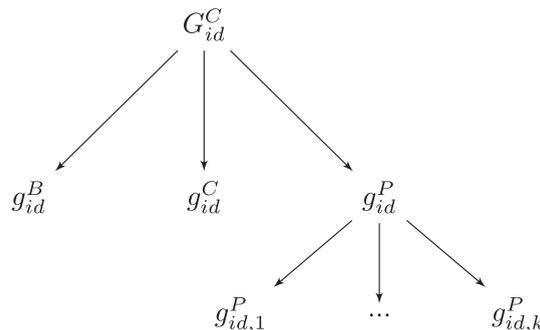


Fig. 4. Decision tree of modal and transfer node choice for users with car and bus availability.

In the same way as for the integrated model with pure modes, in this combined formulation, there is a primary mode choice that considers the generalized costs of each option. Let  $\gamma_{id}^m$  be the probability of choosing mode  $m \in \{B, C, P\}$  to travel between the OD pair  $id$ . Hence,

$$\gamma_{id}^m = \frac{e^{\beta_M U_{id}^m}}{\sum_{m' \in \{B, C, P\}} e^{\beta_M U_{id}^{m'}}} \tag{16}$$

where  $\beta_M$  is the dispersion parameter associated with the logit model at the modal choice level.

The generalized costs in the case of pure car or bus modes may be modeled as in the previous subsection. However, for the combined mode, we have to estimate an expected maximum utility function considering all transfer nodes. Let  $U_{id,k}^P$  be the utility of using the combined mode between the pair  $id$  using transfer node  $k$ . Then, an estimate of the maximum utility among all transfer nodes, which is the representative utility of choosing the combined mode at the upper level of the decision tree, can be determined using a log-sum formulation:

$$U_{id}^P = \frac{1}{\beta_P} \ln \left( \sum_{k \in K_P} e^{\beta_P U_{id,k}^P} \right), \tag{17}$$

where  $\beta_P$  is a measure of correlation in unobserved factors within the parking nest under a hierarchical logit scheme.

Given that a traveler has chosen the combined mode, the conditional probability of choosing  $k$  as a transfer node can be computed as

$$\gamma_{id,k}^P = \frac{e^{\beta_P U_{id,k}^P}}{\sum_{k' \in K_P} e^{\beta_P U_{id,k'}^P}} \tag{18}$$

Finally, the demand between the pair  $id$  on transit and traffic networks are

$$g_{id}^B = G_{id}^B + \gamma_{id}^B \cdot G_{id}^C \tag{19}$$

$$g_{id}^C = \gamma_{id}^C \cdot G_{id}^C \tag{20}$$

respectively, and the total number of travelers who choose the combined mode between the  $id$  pair, while transferring at the  $k$  transfer node, is

$$g_{id,k}^P = \gamma_{id,k}^P \cdot \underbrace{\gamma_{id}^P \cdot G_{id}^C}_{g_{id}^P} \tag{21}$$

Graphically, the modal split for traffic and transit networks in the Integrated Stochastic Equilibrium with combined modes is shown in Fig. 5.

In the integrated model with combined modes, the generalized cost of traveling from  $i$  to  $d$  via mode  $m$  remains the same as in the previous section for the pure modes ( $B, C$ ); however, in the case of the combined option ( $P$ ), the generalized cost or utility function  $U_{id}^P$  depends on the vector  $(U_{id,k}^P)_{k \in K_P}$ . Each  $U_{id,k}^P$  is given by a utility function (one for each parking facility  $k \in K_P$ )  $u_k^P : \mathbb{R}_+^2 \times \mathbb{R}_+^{N \times D} \rightarrow \mathbb{R}$ , where  $u_k^P(t_1, t_2, g_k^P)$  is the generalized cost when the travel time by car from the origin  $i$  to the parking location  $k$  is  $t_1$  and that from the parking location to the destination  $d$  by transit is  $t_2$ . This functions depend as well on the profile of users that utilize the park-and-ride facility located in  $k \in K_P$ ,  $g_k^P = (g_{id,k}^P)_{N \times D}$ , because they can congest this parking facility, which can increase the waiting times of users. In addition to the expected travel times in both car and bus segments of the trip, this function can also depend on other aspects such as parking safety, comfort, accessibility, and parking fares.

With these elements, we are now able to formalize the definition of integrated equilibria with combined modes.

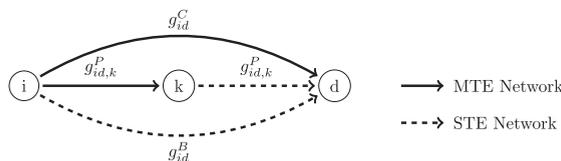


Fig. 5. Modal split on traffic and transit networks with combined modes.

**Definition 4.** Given an urban transportation network  $G = (N, A)$  that contains transit, traffic and parking facilities, a tuple  $(U^*, g^*, v^*, \tau^*)$  is an *Integrated Stochastic Equilibrium with Combined Modes* if flows and expected travel times in the traffic network  $(v^C, \tau^C)$  are an MTE; flows and expected total times in the transit network  $(v^B, \tau^B)$  are an STE; demand  $g^*$  satisfies Eqs. (19)–(21);  $U_{id}^m = u^m(\tau_{id}^m)$  for  $m = \{B, C\}$ ;  $U_{id}^P$  satisfies Eq. (17); and  $U^{P_{id,k}} = u_k^P(\tau_{ik}^C)^*$ ,  $\tau_{kd}^B$ ,  $g_k^{P^*}$  for all  $i \in N$ ,  $d \in D$  and  $k \in K_P$ .

#### 4. Equilibrium algorithms

In this section, we present an efficient algorithm that computes an Integrated Stochastic Equilibrium. The procedure is an iterative method based on the Method of Successive Averages (MSA). At the beginning of each iteration, in addition to computing the travel times and effective frequencies in both networks, demand levels are obtained by calculating respective utility functions for each mode, considering the levels of service obtained in the previous iteration as inputs for the subsequent iteration. The algorithm is designed to reach the equilibrium for the STE and MTE models separately, which computationally can be solved in parallel and not necessarily through a sequential execution. The algorithm compares the resulting vector of flows with respect to the flows obtained in the previous iteration until reaching a predefined convergence criterion based on the similarity of the flows. Before discussing the algorithms to find the described equilibria, in the following section, we present certain considerations that are required to implement the STE module.

##### 4.1. STE implementation details

The STE model contains cross dependencies among the expected travel time  $\tau_{id}$ , conditional probability of boarding a certain bus  $p_a^d$ , effective frequencies  $f_a$ , travel time  $t_a$  and destination flows  $v_{ad}$ . Because of these dependencies, the STE implementation becomes in essence the solution of a fixed-point problem among these values, which are interrelated in Definition 1 and Eq. (1) for flow conservation at each node.

Recalling the definition of our extended network (see Fig. 1), in the case of line nodes, all outgoing arcs from those nodes – alighting and on-board arcs – have infinite frequency in our representation; the same property applies to walking arcs associated with centroids. For these cases, the effective frequencies in Eqs. (4)–(6) are replaced by  $f_\infty$ , and we then take the limit of those equations as  $f_\infty \rightarrow \infty$ , thereby obtaining

$$\tau_{id}^* = \lim_{f_\infty \rightarrow \infty} \frac{1 + \sum_{a \in A_i^+} p_a^d f_\infty (t_a(v^*) + \tau_{jad}^*)}{\sum_{a \in A_i^+} f_\infty p_a^d} = \frac{\sum_{a \in A_i^+} p_a^d (t_a(v^*) + \tau_{jad}^*)}{\sum_{a \in A_i^+} p_a^d}, \quad (22)$$

$$v_{ad}^* = \lim_{f_\infty \rightarrow \infty} x_{id}(v^*) \frac{f_\infty p_a^d}{\sum_{a \in A_i^+} f_\infty p_a^d} = x_{id}(v^*) \frac{p_a^d}{\sum_{a \in A_i^+} p_a^d}, \quad (23)$$

$$p_a^d = \varphi_a(t_a(v^*) + \tau_{jad}^* - \tau_{id}^*). \quad (24)$$

Moreover, stop nodes combine finite and infinite frequencies on their outgoing arcs. To use the equations in Definition 1, we include the following consideration in the algorithm: when computing the equilibrium for destination  $d$ , we remove all boarding arcs from stop nodes adjacent to  $d$  because it is assumed that passengers are rational and that they will not ride another bus; instead, these passenger will walk toward the destination. Conversely, if none of the centroids adjacent to the stop is the destination  $d$ , then walking arcs outgoing from this node are not considered, thus forcing passengers to board another bus to reach their destinations. These modifications are shown in Fig. 6, in which dashed arcs are not considered in each case. Note that after these modifications, for each node, all incoming/outgoing arcs have either infinite or finite frequency, thereby allowing travel times and induced flows to be computed. We denote the resulting set of arcs on the extended network as  $\mathcal{A}$ .

The expected travel times  $\tau_{id}^B$  are obtained by solving a system of linear equations. At each node  $i \in N$  of the extended network and for each destination  $d \in D$ , we obtain the expression for  $\tau_{id}$  of Definition 1 or Eqs. (22)–(24), depending on the type of node. For the case in which  $i$  is a stop node and the destination  $d$  is not adjacent to  $i$ , by rearranging terms of Definition 1, we obtain the following:

$$\sum_{a \in A_i^+} f_a(v) p_a^d (\tau_{id}^B - \tau_{jad}^B) = 1 + \sum_{a \in A_i^+} f_a(v) p_a^d t_a(v), \quad \forall i \in N_S \text{ with } (i, d) \notin \mathcal{A}. \quad (25)$$

If  $i$  is a line, centroid or stop node adjacent to destination  $d$ , then for all outgoing arcs of  $i$ ,  $f_a(v) \rightarrow \infty$  holds, and the redefinition of  $\tau_{id}$  in Eq. (22) applies. By rearranging the terms in Eq. (22), we obtain

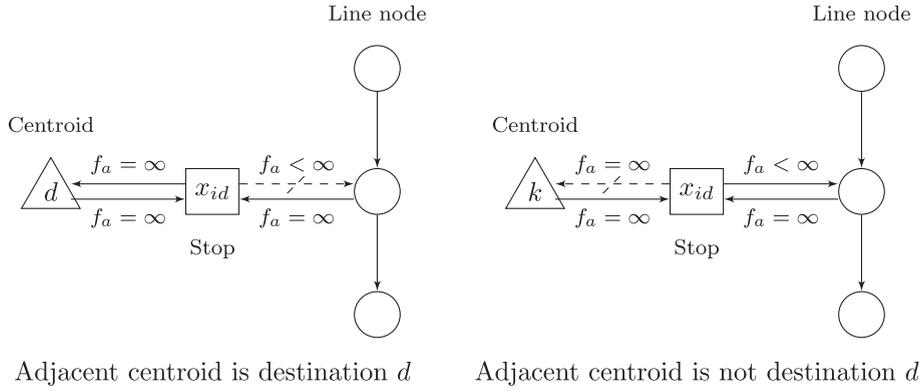


Fig. 6. STE network modifications for stop nodes when computing the equilibrium for destination  $d$ .

$$\sum_{a \in A_i^+} p_a^d (\tau_{id}^B - \tau_{j_{ad}}^B) = \sum_{a \in A_i^+} p_a^d t_a(v), \quad \forall i \in D \cup N_L \cup N_S \quad \text{with } (i, d) \in \mathcal{A}. \quad (26)$$

As shown, Eqs. (25) and (26) form a sparse system of linear equations of size  $|N| \times |N|$ , which can be efficiently solved even for large-scale networks, thereby obtaining the values  $\tau_{id}^B$  for a given flow vector  $v$ .

#### 4.2. Integrated equilibrium algorithm with combined modes

Because the Integrated Stochastic Equilibrium with pure modes is a particular instance of the more general model with combined modes, the implementation of an algorithm to solve the equilibrium in a multimodal network will be based on the more general model. The structure used to implement the algorithm is the following: We use an iterative MSA-based algorithm, in which the expected travel time matrices in each network are first computed; then, the levels of demand in each network are determined using a modal split model; and finally, with the obtained demands, we solve the equilibrium sub-models for each mode. A stop criteria based on the similarity of flow vectors is proposed.

The general integrated equilibrium algorithm is described in Algorithm 1. The stochastic transit and traffic equilibrium modules are described step-by-step in Algorithms 2 and 3, respectively. Note that for the Integrated Stochastic Equilibrium with pure mode implementation, it is sufficient to define the transfer node subset as  $K_P = \emptyset$ .

#### Algorithm 1. Integrated Stochastic Equilibrium with combined modes

- 1: Set initial feasible assignment  $v^{C,0}$  and  $v^{B,0}$  in both networks
- 2: Set  $n \leftarrow 0$ .
- 3: **repeat**
- 4: Set  $n \leftarrow n + 1$ .
- 5: Modal split model:  $g_{id}^{B,n}, g_{id}^{C,n}, g_{id,k}^{P,n}$
- 6: Compute effective frequencies  $f_a^n = f_a(v^{B,n-1})$ .
- 7: Compute travel time  $t_a^n = t_a(v^{n-1})$ , and  $s_a^n = s_a(v^{C,n-1})$
- 8: **Solve transit equilibrium** (see Algorithm 2) with transit demands:

$$g_{id}^{B,n} + \sum_{i \in N} g_{i,d,i}^{P,n} \quad \forall i \in K_P, \quad d \in D$$

$$g_{id}^{B,n} \quad \forall i \in N \setminus K_P, \quad d \in D$$

- 9: **Solve traffic equilibrium** (see Algorithm 3) with traffic demands:

$$g_{id}^{C,n} + \sum_{d \in D} g_{i,d,d}^{P,n} \quad \forall i \in N, \quad d \in K_P$$

$$g_{id}^{C,n} \quad \forall i \in N, \quad d \in D \setminus K_P$$

- 10: **until**  $\frac{\sqrt{\sum_a (\bar{y}_a^{B,n+1} - \bar{y}_a^{B,n})^2}}{\sum_a \bar{y}_a^{B,n}} \leq \epsilon$  and  $\frac{\sqrt{\sum_a (\bar{y}_a^{C,n+1} - \bar{y}_a^{C,n})^2}}{\sum_a \bar{y}_a^{C,n}} \leq \epsilon$ .

**Algorithm 2.** Transit module for Integrated Stochastic Equilibrium

- 
- 1: **for all** destination  $d \in D$  **do**
  - 2: Set  $l \leftarrow 0$ .
  - 3: Compute initial conditional probabilities  $p_a^{d,0} = \varphi_i^d(t_a^n + \tau_{j_{ad}}^{B,n} - \tau_{id}^{B,n})$ .
  - 4: **repeat**
  - 5: Set  $l \leftarrow l + 1$ .
  - 6: Solve system of linear equations for expected travel time

$$\tau_{id}^{B,l} = \frac{1 + \sum_{a \in A_i^+} f_a^n p_a^{d,l-1} \cdot (t_a^n + \tau_{j_{ad}}^{B,l})}{\sum_{a \in A_i^+} f_a^n p_a^{d,l-1}} \quad \forall i \in N_S \quad \text{with } (i, d) \notin \mathcal{A}$$

or,

$$\tau_{id}^{B,l} = \frac{\sum_{a \in A_i^+} p_a^{d,l-1} \cdot (t_a^n + \tau_{j_{ad}}^{B,l})}{\sum_{a \in A_i^+} p_a^{d,l-1}} \quad \forall i \in D \cup N_L \cup N_S \quad \text{with } (i, d) \in \mathcal{A}$$

- 7: Compute conditional probabilities  $p_a^{d,l} = \varphi_i^d(t_a^n + \tau_{j_{ad}}^{B,l} - \tau_{id}^{B,l})$ .
- 8: **until**  $\frac{\|\tau_{id}^{B,l} - \tau_{id}^{B,l-1}\|}{\|\tau_{id}^{B,l}\|} < \epsilon$ .
- 9: Set  $p_a^{d,n} = p_a^{d,l}$
- 10: Compute induced flows

$$\hat{v}_{ad} = x_{id}^B \frac{f_a^n p_a^{d,n}}{\sum_{a \in A_i^+} f_a^n p_a^{d,n}} \quad \forall i \in N_S \quad \text{with } (i, d) \notin \mathcal{A}$$

or,

$$\hat{v}_{ad} = x_{id}^B \frac{p_a^{d,n}}{\sum_{a \in A_i^+} p_a^{d,n}} \quad \forall i \in D, N_L \quad \text{and } \forall i \in N_S \quad \text{with } (i, d) \in \mathcal{A}$$

- 11: **end for**
  - 12: Update transit flow assignment  $v^{B,n} = (1 - \alpha_n)v^{B,n-1} + \alpha_n \hat{v}$ .
- 

**Algorithm 3.** Traffic module for Integrated Stochastic Equilibrium

- 
- 1: **for all** destination  $d \in D$  **do**
  - 2: Set  $l \leftarrow 0$ .
  - 3: Set  $z_{ad}^0 = s_a^n$
  - 4: Set  $\tau_{id}^{C,0} = 0$
  - 5: **repeat**
  - 6: Set  $l \leftarrow l + 1$ .
  - 7: Compute expected travel time  $\tau_{id}^{C,l} = \psi_{id}(z_{ad}^{l-1})$
  - 8: Compute expected travel time by arcs  $z_{ad}^l = s_a^n + \tau_{j_{ad}}^{C,l}$ .
  - 9: **until**  $\frac{\|\tau_{id}^{C,l} - \tau_{id}^{C,l-1}\|}{\|\tau_{id}^{C,l}\|} < \epsilon$ .
  - 10: Compute probabilities

$$q_a^{d,n} = \frac{\partial \psi_{id}(z_{ad}^l)}{\partial z_{ad}^l}$$

- 11: Compute induced flow

$$\hat{v}_{ad} = x_{id}^C \frac{q_a^{d,n}}{\sum_{a \in A_i^+} q_a^{d,n}} \quad \forall i \in D \cup N_S$$

- 12: **end for**
  - 13: Update traffic flow assignment  $v^{C,n} = (1 - \alpha_n)v^{C,n-1} + \alpha_n \hat{v}$ .
-

In some cases, the MSA convergence is not monotonic. This situation occurs because the descent direction may point in a direction such that the norm in certain iterations increases; it could also occur because the MSA step, given by  $\alpha_n$ , is fixed a priori and may exceed the optimal descent weight (Sheffi, 1985). A convergence criterion that is in general monotonically decreasing can be obtained by averaging the flows over the previous  $r$  iterations. In this case, if  $\bar{v}_a^n$  denotes the average flow in the iteration  $n$ ,

$$\bar{v}_a^n = \frac{1}{r} (v_a^n + v_a^{n-1} + \dots + v_a^{n-r+1}) \quad (27)$$

then the convergence criteria may be based on the similarity of flows for the previous  $r$  iterations. For example,

$$\frac{\sqrt{\sum_a (\bar{v}_a^{m+1} - \bar{v}_a^m)^2}}{\sum_a \bar{v}_a^m} \leq \epsilon. \quad (28)$$

The latter is the convergence criterion implemented in Algorithm 1.

Although the latter is the convergence criterion implemented in Algorithm 1, there are other methods which may improve the traditional MSA algorithm performance, and consequently, improve the performance of the Integrated Stochastic Equilibrium algorithm. In particular, Liu et al. (2009) proposed two methods for choosing the step size  $\alpha_n$ . Realizing that the auxiliary flow  $v_a^n$  approaches the solution point of the problem when the iteration number is large, the authors introduced at first place, the Method of Successive Weighted Averages (MSWA) that assigns more weight to the later intermediate flows  $v_a^n$ , instead of a simple equal average as in MSA. However, in both traditional MSA and MSWA, the step size are determined a priori and do not consider any information generated from the algorithm execution. Two disadvantages arise from the latter: (1) the step size may be too large, and leads the next iteration solution lying farther from the optimal solution from the previous one; and (2) when the current solution is close to the optimal solution, the step size may be too small, causing an extremely slow convergence speed. Hence, the authors proposed a self-regulated averaging method, where the step sizes are updated by evaluating a potential function, which could be the distance between the generated auxiliary flow  $v_a^n$  and current solution  $\bar{v}_a^n$ . The self-evaluating method gives more aggressive exploration of the solution space – i.e. large step sizes – when current iteration converge – or the distance between auxiliary flow and current solution are relative small; otherwise, this method gives small step sizes when the solutions diverge. Numerical experiments revealed that self-regulated method is more preferable than MSWA. In the latter, a not known optimal step size is required a priori for each individual problem, for a better convergence speed and accuracy; while in the former, a pseudo-optimal step size is calculated by the own solution procedure.

## 5. Numerical experiments

In this section we analyze various numerical experiments by observing the behavior of the Integrated Stochastic Equilibrium. The pure mode algorithm is implemented in a simple network, focusing first on transit equilibrium and comparing the results among the stochastic model (Cortés et al., 2013) and the deterministic version (Cominetti and Correa, 2001; Cepeda et al., 2006) and then focusing in the integrated equilibrium comparing instances with different levels of stochasticity on each mode.

### 5.1. Stochasticity in the transit equilibrium model

For the sake of completeness, we first illustrate the effect of stochasticity on the transit network. We use the Sioux Falls city coding provided by Bar-Gera (2011) whose layout corresponds to a traffic network, although we use the same configurations for coding the transit mode, which is similar to the behavior of an urban subway system.

The Sioux Falls network contains 24 nodes, each generating and attracting demand. Therefore, each node is simultaneously a centroid and a stop node. There are 360,000 total trips generated per hour in the system. For testing purposes, we designed 4 subway lines around the city, as shown in Fig. 7. Each line  $l$  has a nominal frequency of  $\mu_l = 30$  trains/h and capacity of  $c_l = 1500$  pax/train. The conditional probability function used for this experiment is the following logistic distribution:

$$p_a^d \equiv \varphi_a^d(t_a + \tau_{j_{ad}} - \tau_{id}) = \frac{1}{1 + e^{\theta(t_a + \tau_{j_{ad}} - \tau_{id})}}, \quad \forall a \in \mathcal{A}; \quad \forall d \in \mathcal{D}; \quad \theta \in \mathbb{R}_+. \quad (29)$$

We tested a set of instances for this network, in which the only difference is the parameter  $\theta$  of the conditional probability  $p_a^d$ , which ranges from  $\theta = 30$  (deterministic) to  $\theta = 0.2$  (highly stochastic). Hence, we obtain different stochasticity levels in users' decision-making processes between instances. The formulation of the effective frequency is the same as that used in Cepeda et al. (2006), with the exponent  $\beta = 5.0$ . The convergence criterion is set as  $\epsilon = 10^{-9}$ , and the MSA parameter selected for updating the flow is  $\alpha_n = \frac{1}{n}$ , where  $n$  is the number of the current iteration in the general algorithm. The transit equilibrium is obtained after approximately 250 iterations for each instance.

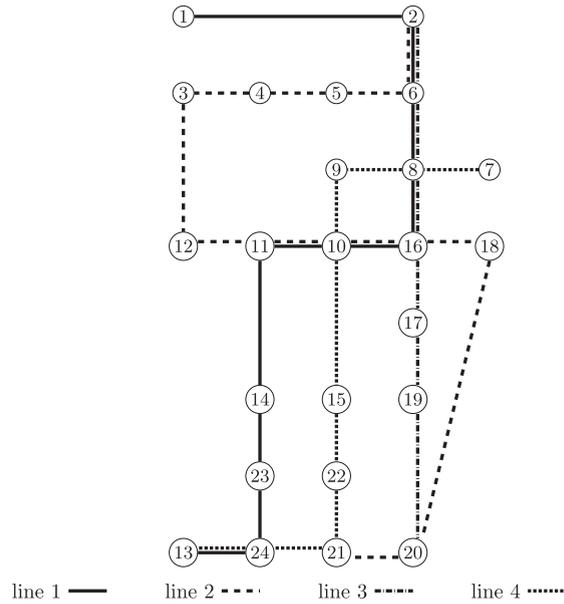


Fig. 7. Design of transit lines in Sioux Falls.

To analyze the results obtained using the transit equilibrium model, it is possible to perform comparisons between the deterministic and stochastic instances if we define adequate dispersion indicators in equilibrium flows over the network. In principle, in deterministic instances, users choose the minimum hyperpath, which causes the arcs or segments contained in this strategy to have flows  $v_{ad}^* > 0$ , whereas for arcs that belong to suboptimal hyperpaths,  $v_{ad}^* = 0$ . Then, we define the *level of network usage*, denoted  $\Psi_d^B$ , as the ratio between the number of line segments used in the network to reach the destination  $d$  and the total number of line segments available in the network;  $\bar{\Psi}^B$  is the average level of network usage over all destinations. A *used segment* is such that  $v_{ad}^* > 0$ . However, because the transit equilibrium algorithm is a numerical implementation, we consider that a used segment is such that  $v_{ad}^* > \xi \approx 0$ . In the case of Sioux Falls,  $\xi = 0.5$  [pax/h]. As expected, a reduction in  $\theta$  causes an increase in the average level of network usage over all destinations  $\bar{\Psi}^B$ . In Table 1 we display the values of  $\bar{\Psi}^B$  and we illustrate the level of network usage for destination  $d = 10$  between deterministic and highly stochastic instances with Figs. 8 and 9.

5.2. Integrated Stochastic Equilibrium with pure modes

In this experiment, we analyzed the Integrated Stochastic Equilibrium including pure modes only; a private car transport network of Sioux Falls city is used (Bar-Gera, 2011), which contains 24 centroids, 76 unidirectional arcs, and total demand of 360,000 trips/h. Moreover, the public transport network and simulation parameters are identical to those used in testing of the transit equilibrium algorithm, as described in Section 5.1.

The objective of this experiment is to measure the impact of the stochasticity in both the traffic and the transit networks. For this purpose, a logistic formulation is used in the transit model for the conditional probability  $p_a$  (Eq. (29)), varying the parameter  $\theta$ . In the case of private cars, a log-sum formulation was chosen for computing the expected minimum travel time  $\tau_{id}^C$ , in which case the parameter  $\beta$  is varied to obtain different levels of stochasticity. Note that the modal utility functions are based only on expected travel times for each network, i.e.,  $U_{id}^C = -\tau_{id}^C$  and  $U_{id}^B = -\tau_{id}^B$ .

The general results obtained using the Integrated Stochastic Equilibrium algorithm with pure modes for the Sioux Falls network are shown in Tables 2 and 3. These tables summarize the modal split for users with car and transit availability and network usage indicators.

As shown in Table 2, increasing stochasticity in the transit system and maintaining a deterministic formulation in the traffic formulation causes the users to tend to choose a car in higher proportions because the expected travel time computed

Table 1  
Average level of Sioux Falls network usage.

$\theta$	30.0	15.0	8.0	4.0	2.0	1.0	0.5	0.3	0.2
$\bar{\Psi}^B$	0.44	0.44	0.46	0.49	0.65	0.93	0.96	0.96	0.96

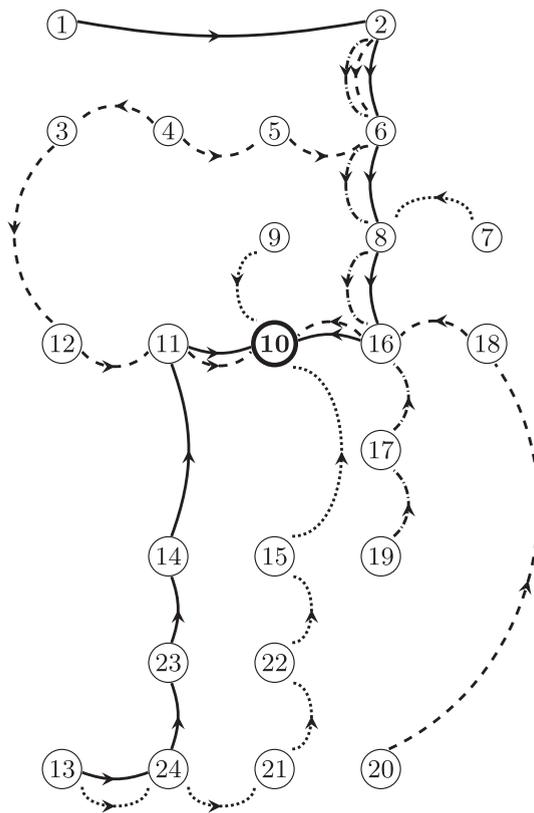


Fig. 8. Transit segments used in the deterministic transit equilibrium for Sioux Falls instance.  $\theta = 30$  and  $\psi_{10}^g = 0.46$ .

in equilibrium on transit integrates suboptimal strategies, thereby increasing these values. Hence, users who have modal choice availability tend to choose the mode that is more predictable in relation to travel times. Furthermore, the same results indicate that transit network usage strongly increases as the stochasticity increases, from 0.43 in the deterministic instance to 0.96 in the fully stochastic instance.

A similar analysis can be performed by observing the results in Table 3 when increasing stochasticity in traffic system and with a deterministic formulation remaining in transit formulation. In this case, increasing the stochasticity for traffic causes a higher proportion of users to choose public transport due to less variability in expected travel times; however, changes in the modal split are lower compared to the previous case. On the other hand, stochasticity in traffic increases the use of arcs in the private transport network such that, on average, the flow for a given destination is dispersed through all available arcs.

### 6. Integrated Stochastic Equilibrium with park-and-ride in a real network

In this section we implement the Integrated Stochastic Equilibrium with the combined modes algorithm and we perform/report a study on the effect of installing park-and-ride facilities in a real network representation of the city of Iquique in the north of Chile.

The real transit network contains 72 centroids, 485 bus stops and 2118 unidirectional transit line segments. The network belongs to a weekday morning peak period, calibrated for the year 1998, with 5449 trips/h being performed during this period. The size of the transit extended network for this city is 2711 nodes – including centroids, stops and line nodes – and 7144 arcs – including walking, boarding, alighting and on-board arcs.

On the other hand, the traffic network includes 72 centroids, 485 intersection nodes and 2180 unidirectional road arcs, which are identical to those contained in the transit network using the same road infrastructure, for a total of 10,646 trips/h.

The Integrated Stochastic Equilibrium with combined modes requires modal utility functions and parameters to be defined to obtain the modal share of users that have car, bus and combined mode availability. The network coding, modal utility functions and various calibration parameters for the modal split model for Iquique were provided by the Ministry of Transport and Telecommunications of the Chilean Government. We will use the following utility function,  $u^c$ , which is used in equilibrium to calculate for each  $id$  the utility  $U_{id}^c$  of a user traveling from  $i$  to  $d$  using a car:

$$u^c(\tau_{id}^c) = \theta_c + \theta_{tgen} \cdot \tau_{id}^c + \theta_{cost} \cdot \frac{\tau_{id}^c c_{unit}}{I} \tag{30}$$

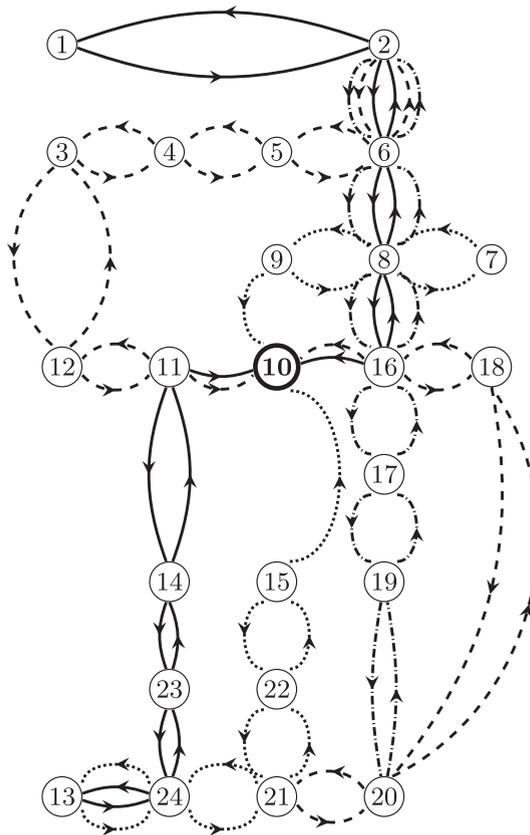


Fig. 9. Transit segments used in the stochastic transit equilibrium for Sioux Falls instance.  $\theta = 0.5$  and  $\Psi_{10}^B = 0.91$ .

Table 2

Results from the Integrated Stochastic Equilibrium with pure modes for the Sioux Falls network, stochasticity in transit.

$\theta$ (Transit)	$\beta$ (Traffic)	Car share (%)	Bus share (%)	$\bar{\psi}^B$	$\bar{\psi}^C$
30.0	12.0	67.8	32.2	0.43	0.48
15.0	12.0	67.8	32.2	0.44	0.48
8.0	12.0	67.8	32.2	0.44	0.48
4.0	12.0	67.8	32.2	0.46	0.48
2.0	12.0	68.2	31.8	0.49	0.48
1.0	12.0	69.4	30.6	0.65	0.49
0.5	12.0	72.3	27.7	0.93	0.49
0.3	12.0	76.9	23.1	0.96	0.50
0.2	12.0	81.7	18.3	0.96	0.52

Table 3

Results from the Integrated Stochastic Equilibrium with pure modes for the Sioux Falls network, stochasticity in traffic.

$\theta$ (Transit)	$\beta$ (Traffic)	Car share (%)	Bus share (%)	$\bar{\psi}^B$	$\bar{\psi}^C$
30.0	12.0	67.8	32.2	0.43	0.48
30.0	6.0	67.9	32.1	0.43	0.49
30.0	3.0	68.0	32.0	0.43	0.49
30.0	1.5	68.0	32.0	0.43	0.57
30.0	1.0	67.8	32.2	0.43	0.69
30.0	0.5	66.6	33.4	0.44	0.98
30.0	0.4	65.8	34.2	0.44	1.00
30.0	0.35	64.9	35.1	0.44	1.00

where

$\theta_C$	car modal constant [utility]
$\theta_{tgen}$	generalized time parameter [utility/min]
$\theta_{cost}$	monetary cost parameter [utility/\$]
$C_{unit}$	car use monetary cost per unit time [\$/min]
$I$	income level [\$]

In addition, the utility function for passengers using a bus,  $u^B$ , which is used in equilibrium to calculate for each  $id$  the utility  $U_{id}^B$  of a user traveling from  $i$  to  $d$  using the bus mode, is:

$$u^B(\tau_{id}^B) = \theta_B + \theta_{tgen} \cdot \tau_{id}^B + \theta_{cost} \cdot \frac{C_{bus}}{I} \quad (31)$$

where

$\theta_B$	bus modal constant [utility]
$C_{bus}$	bus fare [\$]

The remaining parameters for the bus utility function are the same as those defined in the case of the car in Eq. (30), and to increase the performance of the simulation, most of the parameters for Iquique, shown in Table 4, were previously calibrated. We use  $\beta_M = 1$  and  $\beta_P = 1.41$  for the modal choice and parking choice parameters.

The combined mode representative utility  $U_{id}^P$  using a log-sum formulation of Eq. (17) is required. First, we need to compute the value of the utility corresponding to the combined mode, in which the interchange between car and bus occurs at parking facility  $k \in K_P$ ,  $U_{id,k}^P$ . For this purpose, we assume that this utility has a functional cost and time structure similar to the sum of utilities of choosing each mode in their respective section of the trip. In addition, we add a constant  $\theta_p$  that represents the modal interchange disutility at the parking lot, which reflects other conditions not included as variables in our model such as comfort, security, accessibility, and infrastructure. In the case of Iquique, this constant was not previously calibrated; therefore, a consistent value is assumed to facilitate the simulation scenario. Moreover, we will add to the combined mode utility the fare of parking a car  $c_{park}$  as well as the queuing time caused by other vehicles when entering the parking lot. The latter term, denoted by  $W^k$ , is computed as follows: consider that the parking entrance is a queue of type  $M/M/r$ , where arrivals and service in the parking facility are assumed to be Markovian with arrival rates  $\lambda_k$  together with  $r$  available servers, each of which having a service rate  $\mu_k$ . Then, the average waiting time in a queue with these characteristics can be approximated by Larson and Odoni (1981)

$$W^k \approx \frac{\frac{\lambda_k}{(r_k \mu_k)^2}}{1 - \frac{\lambda_k}{r_k \mu_k}} + \frac{1}{\mu_k} \quad (32)$$

Eq. (32) has an implicit capacity constraint because it is only valid when  $\frac{\lambda_k}{r_k \mu_k} < 1$ . In the algorithm implementation, the arrival rate to the parking lot is equal to the sum over all the  $id$  pairs of the number of users who travel between origin  $i$  and destination  $d$  using the combined mode  $P$  and choosing as transfer node the centroid that represents the parking lot, i.e.,

$$\lambda_k = \sum_{(i,d) \in N \times D} g_{id,k}^P$$

Thus, the utility of choosing the combined mode using parking lot  $k$  as a modal interchange location is as follows:

$$u_k^P(\tau_{ik}^C, \tau_{kd}^B, g_k^P) = \theta_P + \left( \tau_{ik}^C + W^k + \tau_{kd}^B \right) \theta_{tgen} + \theta_{cost} \frac{\left( \tau_{ik}^C C_{unit} + c_{park}^k + C_{bus} \right)}{I}$$

**Table 4**  
Modal utility parameters for Iquique, year 1998.

Parameter	Mode		
	Car	Bus	Combined
$\theta_m$	0.502	0.278	-1.500 <sup>a</sup>
$\theta_{tgen}$	-0.023	-0.023	-0.023
$\theta_{cost}$	-43.800	-43.800	-43.800

<sup>a</sup> Not a calibrated parameter. Defined for simulation purposes.

where the dependence on  $g_k^p$  is implicit in the calculation of  $W^k$ . To calculate  $\lambda_k$  (and thus  $W^k$ ) on each iteration of the algorithm, we use the corresponding values  $g_{id,k}^p$  obtained from the modal split of the previous iteration. To assure that the capacity constraint is satisfied, we assign a value of  $\infty$  to  $W^k$ , whenever  $\frac{\lambda_k}{r_k \mu_k} \geq 1$ .

### 6.1. Numerical experiments

In our experiments, we sequentially incorporate park-and-ride facilities (P&Rf) in specific locations of our network by the following rule: we first run a free-of-parking instance of our model as a benchmark case. Next, for each centroid we run an instance of the model with the centroid being the only available P&Rf location. From this set of runs, we select a single P&Rf location, considering the instance that provided the highest occupancy of the P&Rf among all instances in the set. Then, fixing the previously selected location for the P&Rf, we perform a new set of runs of the model, one instance for each remaining centroid. In each instance, there are two available P&Rf locations: the centroid under scrutiny and the previously fixed centroid. We select a second P&Rf considering the instance that provided the highest occupancy of the two P&Rfs. We continue with this procedure of selection of P&Rf until five locations for P&Rfs are selected. This is one possible criterion, chosen arbitrarily to explore how our model responds to the addition of P&R facilities.

The results of our experiments are displayed in Table 5. We show the selected location of the P&Rf on each step of our experiment along with the modal share (car, bus and combined modes) and the occupancy of each of the P&Rfs for the equilibria obtained by the Integrated Equilibrium Model with combined modes for Iquique. The modal share in equilibrium takes into account only the trips of users with car availability.

The modal share in equilibrium obtained for the last step of our experiment, considering users with car availability only, shows that around 63% of the trips are made by car and 31% by bus as pure modes. We observe that the combined mode attracts approximately 6% of the trips starting at 5.1% when only one P&Rf is available, with decreasing increments at each P&Rf addition. The share of the combined mode stabilizes at the addition of the third P&Rf. The increment on the combined mode share comes from a reduction of car participation and not from trips that utilize only bus.

In Fig. 11 we illustrate the percentage of trips that use P&R per OD pair when three P&Rf are opened (namely at nodes 10, 15 and 34). Nodes between 52 and 69 represent the isolated suburb of Alto Hospicio (AH), while nodes between 1 and 20 roughly represent downtown Iquique (refer to Fig. 10).

We observe that internal trips in AH (upper right corner of Fig. 11) do not use the combined mode, which is very reasonable since there are no P&Rf located inside AH and at the same time these are not incorporated in our exercise probably because the attraction of trips in the modeled period is very small as it corresponds mostly to a residential area. The upper left portion of the matrix shows that there is relatively high use of P&R in trips that start at AH (52–69) with downtown destination (6–14). The combined mode share for these trips (the only mode that occupies P&R) is at least 6% and in most of them is even greater than 8%. In the lower and left strips of the matrix we see that trips with downtown origin or destination have, in general, higher usage of P&R. Finally, some trips from AH with destination in the middle part of the city appear to have relatively high usage of P&R.

In Table 6 we show the variation of expected travel time in the combined mode for selected OD pairs (refer to Fig. 10). The node with the highest generation of trips is 28 while the node with highest attraction of trips is 9, thus we include the OD pair (28, 9). OD pairs (60, 6) and (47, 6) are representatives of AH-downtown and south-downtown trips (respectively) with relatively high usage of P&R (more than 8%). Finally, we include two OD pairs: the OD pair (1, 38) because among the OD pairs going from north to south, it presents a high share of combined mode; and the pair (7, 52) whose origin is near the shoreline, its destination is in AH and the transfer is conducted at intermediate points.

We see that the addition of the first three P&Rfs reduces considerably the travel time on four out the five selected OD pairs. For the outlier OD pair (7, 52) travel time does not significantly change since this pair makes almost exclusive use of transfer node 15 (see Table 7 below). Small variations on travel time after the addition of the fourth and fifth transfer nodes are not significant (less than one minute). However, we may point out that the increase in expected travel time for the pair (28, 9) after the addition of transfer node 13 may be due to congestion provoked by the attraction of cars to this node, which is in between nodes 10 and 15, which are used as transfer nodes in this OD pair (see Table 7 below). The same

**Table 5**  
Modal split and park occupancy with sequential selection of P&R facilities.

P&Rfs	Modal split (%)			P&R occupancy (%)
	Car	Bus	Combined	
∅	66.9	33.1	–	–
15	63.5	31.5	5.1	94.9
15–34	63.2	31.3	5.6	84.8–33.4
15–34–10	62.8	31.1	6.1	45.9–31.5–51.5
15–34–10–13	62.7	31.1	6.2	32.1–30.0–40.1–30.2
15–34–10–13–37	62.6	31.0	6.3	31.5–19.4–39.9–29.6–14.1

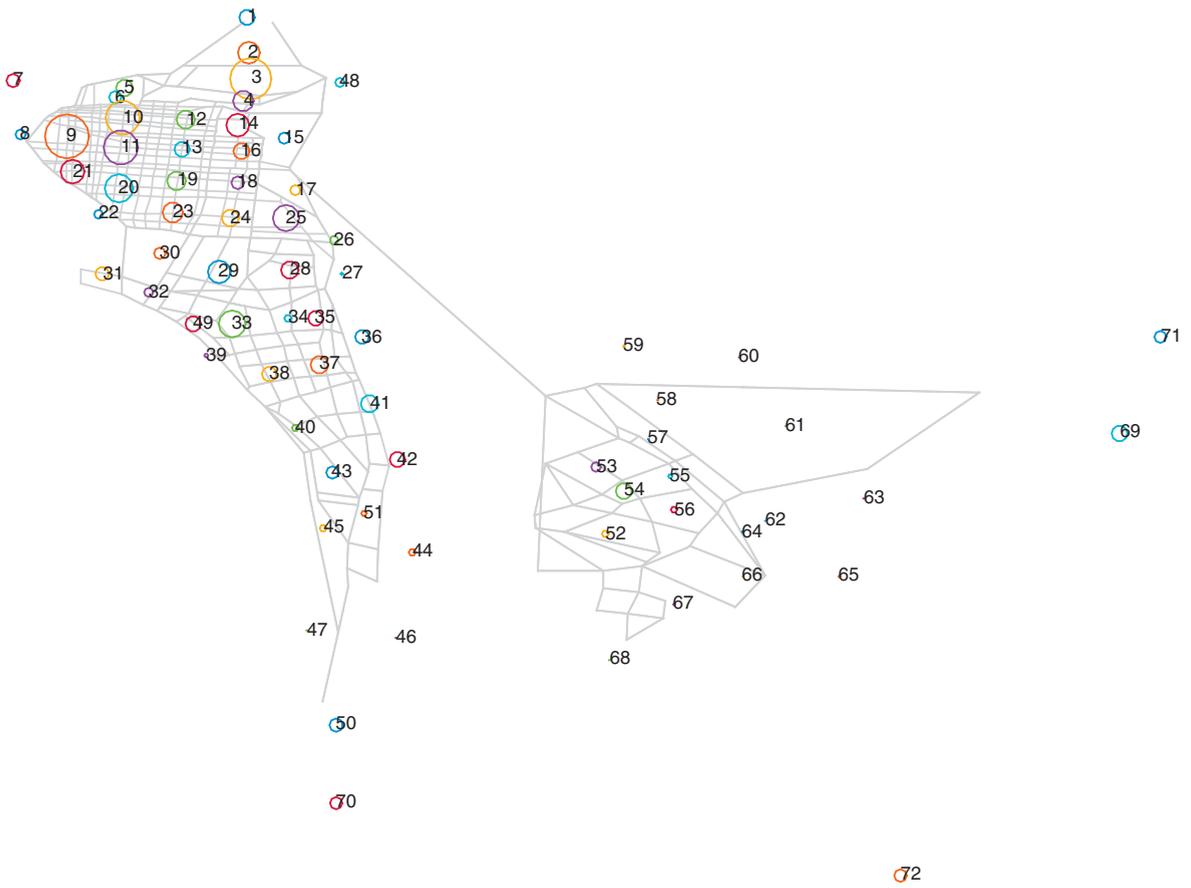


Fig. 10. Iquique traffic network and centroids (circles represent the demand at each centroid).

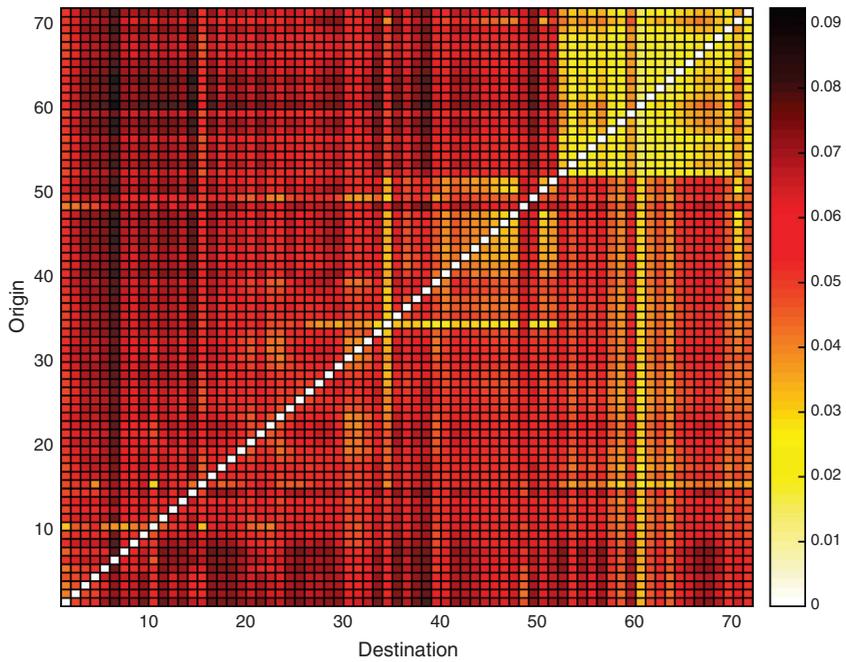


Fig. 11. P&R occupancy per OD pair.

**Table 6**  
Expected travel time using combined mode by OD pair.

P&Rfs	OD pair				
	(28, 9)	(60, 6)	(47, 6)	(1, 38)	(7, 52)
15	26.6	35.8	44.6	39.1	33.3
15–34	25.8	34.9	43.6	20.6	32.6
15–34–10	18.3	19.4	25.9	20.6	33.0
15–34–10–13	19.0	19.4	25.8	20.6	33.2
15–34–10–13–37	19.0	19.4	25.8	21.3	33.2

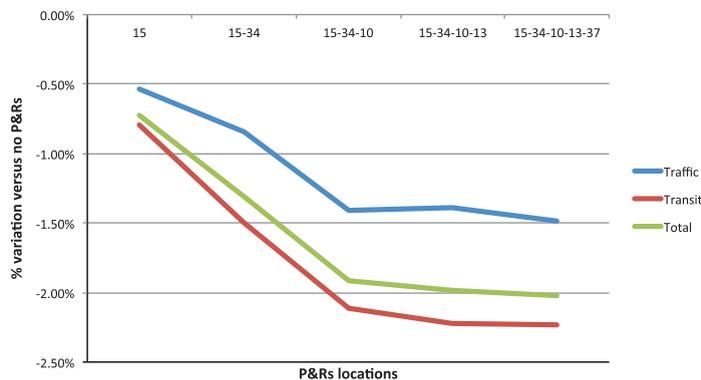
**Table 7**  
Level of service by OD pair.

Orig.	Dest.	With 3 P&Rfs (15–34–10)					$\tau_{ik}^C$	$\tau_{kd}^B$	$W^k$	Combined/car travel time ratio
		$\tau_{id}^C$	$\tau_{id}^B$	P&R share (%)	Parking node <i>k</i>					
28	3	13.6	59.3	7.0	15 (59%)	7.4	19.8	0.17	2.0	
					10 (41%)	11.3	16.3	0.18	–	
28	9	11.5	38.0	7.1	15 (11%)	7.4	17.0	0.17	–	
					10 (89%)	11.3	6.0	0.18	1.5	
60	9	16.7	76.3	8.5	15 (11%)	7.4	17.0	0.17	–	
					10 (89%)	11.3	6.0	0.18	1	
60	6	16.3	89.6	9.2	10 (99%)	15.8	3.3	0.18	1.2	
60	14	15.3	80.0	9.0	15 (99%)	11.3	7.4	0.17	1.2	
60	38	16.6	87.5	8.6	34 (99%)	14.6	7.9	0.15	1.4	
40	6	17.5	72.0	8.6	10 (99%)	17.1	3.3	0.18	1.2	
47	6	22.8	72.8	8.4	10 (99%)	22.4	3.3	0.18	1.1	
1	38	14.4	75.8	8.4	34 (99%)	12.5	7.9	0.15	1.4	
7	52	21.2	67.4	7.6	15 (82%)	5.4	26.5	0.17	1.5	
					10 (18%)	1.9	35.3	0.18	–	

phenomenon may be occurring when adding a P&Rf at node 37, which slightly increases travel time in the OD pair (1, 38) (refer to Fig. 10).

In Table 7 we report the level of service in terms of total expected travel time on each mode of selected OD pairs (refer to Fig. 10). In addition to the OD pairs displayed in Table 6, we include the OD pair with the highest demand, which is (28, 3). Then we include three more representative OD pairs with origin in AH and downtown destination with relatively high usage of P&R (more than 8%). Similarly, we include one extra representative OD pair with origin in the southern part of the city and destination located at downtown, with high usage of P&R. We list the expected travel time on each mode, separating the travel time by car, by bus, and by using the combined mode, along with the occupied P&Rf.

We observe that in all the listed OD pairs, the best (in terms of travel time) P&R option is worse than the car option but far better than the pure bus mode. In half of the listed OD pairs, the expected travel time on the combined mode is comparable with the pure car mode travel time (less or equal to 1.2 times). From Fig. 10 we see that in these OD pairs, the P&R facilities are located fairly close to the destination -relative to the length of the trip-, which implies that the transit mode is mainly used as a local feeder service. In the remaining OD pairs, with the exception of (28, 3), travel time in the combined mode is less or equal to 1.5 times the travel time in the pure car mode. In general, in all trips where the transfer node is near the



**Fig. 12.** Total travel time variation of each mode.

destination node, travel time by car in the combined mode is strictly less than travel time to destination by car, therefore, relative performance of combined mode strongly depends on transit level of service provided. This last observation is highlighted in the OD pair (28, 3) where it takes 20 min in equilibrium to get from the transfer node 15 to the destination node 3. The behavior of travelers using P&R in the OD pair (7, 52) is inverted with respect to the previously discussed cases, since car in the combined mode is used as an access to transit service leaving the larger portion of the trip conducted by bus.

Finally, in Fig. 12 we show how the total expected travel time of the system varies with the addition of each new P&R facility. In this figure we graph the variation of total travel time with respect to the no P&R facilities case, in total and separated by mode. We see that the addition of the first three P&R facilities reduces total travel time on the transit mode in about 2.2% and is further reduced to 2.3%; while total travel time by car is reduced on 1.4% by the addition of the first three facilities and up to a reduction of 1.5%. Total reduction on travel time in this experiment is about 2% of total travel time in the system without P&R facilities.

## 7. Conclusions

In this paper, we presented an integrated traffic-transit stochastic equilibrium model based on state-of-the-art equilibrium models for transit and traffic, including STE (Cortés et al., 2013) and MTE (Baillon and Cominetti, 2008). The model explicitly includes the uncertainty that drivers and passengers experience while they are choosing routes (strategies) until completing their trip, recognizing that within the population, a lack of knowledge about conditions and physical characteristics of the networks cause different perceptions among people. This model also allows us to create an Integrated Stochastic Equilibrium scheme that captures the interaction between the two modes, recognizing that both cars and buses may share the same road infrastructure in many urban areas; the model is therefore, able to incorporate the phenomenon of traffic and passenger congestion. Moreover, the integrated equilibrium model reflects the interaction at the demand and modal share levels because part of the population has the option to choose between the two modes or even combine them in a park-and-ride scheme. This integrated scheme apart from considering stochasticity, incorporates, and combination of modes.

We have shown how to apply the stochastic transit equilibrium model to real networks while considering their specific characteristics. The major issue concerns the equilibrium conditions of Definition 1 for the cases of nodes that are adjacent to arcs that have infinite frequency. A second issue is how to construct an extended network that results from the provided operation pattern of the transit system, incorporating stop nodes, alighting and boarding arcs and access arcs that connect centroids to bus stop nodes.

We provided an algorithm that can be used to obtain the Integrated Stochastic Equilibrium (for both pure and combined modes), and we presented numerical experiments on real networks. In these experiments, we observed, as expected, that an increase in stochasticity causes greater dispersion of equilibrium flows and an increase in the expected travel time. The implementation on a large network, including combined modes, obtains equilibrium flows in a fairly reasonable execution time. In terms of policy issues and implications, we strongly believe that even for strategic-level modeling, the stochastic aspect of human behavior when making daily travel decisions in real urban scenarios plays a fundamental role and should be considered when policy makers decide to invest in various transport projects and plans. The development of an efficient tool for modeling all of these aspects under an equilibrium scheme considering public and private transport in an integrated scheme can then make a major difference in such important decisions.

The general algorithm proposed to solve for the Integrated Stochastic Equilibrium consolidates the formulations and methods of each model individually. One of the major advantages in this sense is that partial equilibria are solved for simultaneously rather than sequentially, which results in an important computational time reduction and consistency in the final results.

Note that although we propose a hierarchical logit modal share formulation between the two systems, the Integrated Stochastic Equilibrium model is sufficiently flexible to integrate other modal share proposals without losing the structural elements of the final joint stochastic equilibria. This issue could be explored as a next step of this research together with other research topics such as the appropriate formulation of the conditional probability of boarding. Other issues to analyze in the future are related to convergence and to the uniqueness of solutions for the integrated approach built from results already proven for the separate cases.

In algorithmic terms, the increase in convergence speed to an equilibrium using methods different from MSA is another topic that requires further study. For example, applying MSWA or optimizing the step value by other numerical methods based on linear search or Newton's method, may result in a faster execution of the algorithm.

Practical applications of our model include the implementation in a real and updated network of the city of Santiago-Chile for planning and policy purposes (including, for instance, optimal location of P&R facilities). This implies a great effort in coding and calibration issues. In the future, we expect to have a planning tool that could be used by policy makers, not only in Santiago-Chile but also in other potential cities abroad, to make proper transport planning decisions for future investments.

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