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Dynamic of cyclic automata over $\mathbb{Z}^2$ ☆

Martín Matamala*, Eduardo Moreno

Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Centro de Modelamiento Matemático, Universidad de Chile, UMR 2071, UCHILE-CNRS, Casilla 170-3, Correo 3, Santiago, Chile

Abstract

Let K be the two-dimensional grid. Let q be an integer greater than 1 and let $Q=\{0,\dots,q-1\}$. Let $s:Q\to Q$ be defined by $s(\alpha)=(\alpha+1)\bmod q$, $\forall\alpha\in Q$.

In this work we study the following dynamic F on $Q^{\mathbb{Z}^2}$. For $x\in Q^{\mathbb{Z}^2}$ we define $F_v(x)=s(x_v)$ if the state $s(x_v)$ appears in one of the four neighbors of v in K and $F_v(x)=x_v$ otherwise.

For $x\in Q^{\mathbb{Z}^2}$, such that $\{v\in\mathbb{Z}^2 : x_v\neq 0\}$ is finite we prove that there exists a finite family of cycles such that the period of every vertex of K divides the lcm of their lengths. This is a generalization of the same result known for finite graphs. Moreover, we show that this upper bound is sharp. We prove that for every $n\geqslant 1$ and every collection $k_1,\dots,k_n$ of non-negative integers there exists $y^n\in Q^{\mathbb{Z}^2}$ such that $|\{v\in\mathbb{Z}^2 : y^n_v\neq 0\}|=O(k_1^2+\cdots+k_n^2)$ and the period of the vertex (0,0) is $p\cdot\operatorname{lcm}\{k_1,\dots,k_n\}$, for some even integer p.

1. Introduction

Let $G=(V,E)$ be a non-directed graph and let $Q=\{0,1,\dots,q-1\}$ be where q is a non-negative integer with $q\geqslant 2$. A Q-state on G is a vector $(x_v : v\in V)$ such that $x_v\in Q$ for all $v\in V$.

In this work, we study the dynamical system (Q^V,F) where F is given as follows. Let x be a Q-state of G. For every $v\in V$, the state $F_v(x)$ is given according to the next-cyclic neighbor (NCN) rule: if the state $s(x_v):=x_v+1\bmod q$ is the state of a neighbor of v then the new state of x_v is $s(x_v)$. Otherwise, the state of vertex v does not change.

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* Corresponding author. Tel.: +56-2678-45-55; fax: +56-2688-38-21.
E-mail addresses: mmatamal@dim.uchile.cl (M. Matamala), emoreno@dim.uchile.cl (E. Moreno).

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The convergence to a global state where all the vertices have the same state was studied in [5]. There, the initial state is defined by assigning to each vertex a state chosen uniformly random from the set Q. This model was also previously studied in [1,2] as a Markov process in continuous time with stochastical evolution, that is, the time difference between two consecutive steps is a random variable with exponential distribution. In both works the model studied was over unidimensional and bi-dimensional grids.

For $q=2$ this process becomes the voting model, studied in [3,12]. For further references see [6–11].

In [13] this dynamic was studied in finite graphs, when all the vertices update their states synchronously in discrete times. Obviously, for finite graphs the sequence of global states generated by the dynamic eventually becomes periodic. The main result in [13] is that the length of this period is upper bounded by the lcm of the lengths of some *special* cycles of the graph. Moreover, for every non-negative integer n it is constructed a graph G_n with $p(n)$ vertices, for some polynomial p, with n special cycles of lengths $4, 8, \ldots, 4n$ and such that the period of the dynamic is at least $4 \cdot \mathrm{lcm}\{1, 2, \ldots, n\}$. Therefore, for general graphs the upper bound is the best possible.

In this work we study the NCN rule in the two-dimensional lattice K. This is an infinite graph where each vertex has four neighbors: left, right, up and down. Given an Q-state on K the state p of a vertex u changes if and only if some of its neighbor v has the state $s(p)$. Therefore, the set of all the edges uv connecting vertices in states p and p' with $p=s(p')$ or $p'=s(p)$ completely determines the next states of all the vertices in K. For technical reasons the edges connecting vertices with the same state are also relevant. That leads us to the following definition.

The skeleton of K with respect to an Q-state x is the subgraph of K defined as follows. Its set of vertices is the set of vertices of K and its edges are those edges of K connecting vertices in the states p and p' with $p=p'$ or $p=s(p')$ or $p'=s(p)$.

In Section 3, we study the dynamic of NCN rule when the skeleton is the whole graph K. For $q \geqslant 5$ we prove that the period at any vertex is 1, i.e., after a finite time the state of the vertex remains constant. For $q=4$ we show that either the period of all vertices is 1 or the period of all vertices is four. We also prove that when $q=3$ the dynamic is richer: for every even integer $t \geqslant 14$ there exists a Q-state on K with at most t states different from zero such that the period of every vertex is exactly t.

In Section 4, we present the main contribution of the paper. We prove that for every $q \geqslant 4$ and for every non-negative integer n there exists a Q-state on K such that there is a vertex that has period at least $4 \cdot \mathrm{lcm}\{1, 2, \ldots, n\}$. Moreover, the Q-state has at most $p(n)$ states different from zero, where p is a polynomial function of n. To prove this result we embed a construction given in [13] into the graph K. We also prove that the upper bound for the period in finite graphs remains valid in the infinite graph K when the Q-state has a finite number of non-quiescent states.

In Section 2, we present the known dynamical results about NCN dynamic in finite graphs which are necessary to present our results.

2. Definition of the problem

We give the main definitions used in NCN. The notation used throughout the paper is consistent with those from the book [4].

Let G be a graph and let x be a Q-state on G. The *skeleton* $Es(G,x)=(V,E(x))$ of x on G is the spanning subgraph of G whose set of edges is defined as follows. An edge $e=uv$ belongs to $E(x)$ if and only if the states x_u and x_v are *adjacent* in Q i.e., $x_u=x_v$ or $x_u=s(x_v)$ or $x_v=s(x_u)$. The *skeleton* of a Q-state x on K is denote $Es(x)$. For a given Q-state x on K we define the *skeleton* of x (on K) as the spanning subgraph $Es(x)=(\mathbb{Z}^2,E(x))$ of K whose set of edges is defined as follows. An edge $e=uv$ belongs to $E(x)$ if and only if the states x_u and x_v are *adjacent* in Q that is, $x_u=x_v$ or $x_u=s(x_v)$ or $x_v=s(x_u)$.

We say that a Q-state x on K has finite support if the cardinality of the set

$$\text{supp}(x):=\{v\in\mathbb{Z}^2 : x_v\neq 0\}$$

is finite. In the sequel a Q-state will refer to a Q-state of finite support.

For a Q-state of finite support the set of edges not in the skeleton of x is finite. Moreover, if $F(x)$ is the Q-state obtained from x by applying the NCN rule then $Es(x)\subseteq Es(F(x))$ (see [13, Lemma 1]). Hence, there is a time t^* such that $Es(F^{t^*+i}(x))=Es(F^{t^*}(x))\ \forall i$ i.e., after time t^* the skeleton becomes stable.

We need some definitions before to describe the generalization of the known dynamical properties of NCN from finite to infinite graphs. Let $G=(V,E)$ be a graph. A *walk* P in G is a sequence $v_0,\dots,v_k$ of vertices of G such that v_i is adjacent with v_{i+1} for $i=0,\dots,k-1$. It is a *closed walk* if $v_0=v_k$. The *length* of P is $k-1$ and it is denoted by $L(P)$. If all the vertices v_i are different then P is called a *path*. If we add the edge v_0v_k to a path $v_0,\dots,v_k$ we obtain a *cycle*.

For two states α and β adjacent in Q, we define the jump $\mu(\alpha,\beta)$ by $\mu(\alpha,\beta)=1$ if $s(\alpha)=\beta$, $\mu(\alpha,\beta)=-1$ if $s(\beta)=\alpha$ and $\mu(\alpha,\beta)=0$ otherwise.

Let x be a Q-state on G. Let $P=(v_0,\dots,v_k)$ be a walk in the skeleton $Es(G,x)$. The jump $J(P,x)$ of P on x is the sum of the jumps $\mu(x_{v_i},x_{v_{i+1}})$, for $i=0,\dots,k-1$. In this definition the order of the vertices in P is relevant. Indeed, if $-P=(v_k,\dots,v_0)$ then $J(-P,x)=-J(P,x)$. Notice that if P is a closed walk then q divides $J(P,x)$.

For an integer $L\geqslant 1$ and a vertex $v\in V$ let $a(v,L,x)$ the maximum jump among all the jumps (on x) of walks starting in v and having lengths less or equal than L. Let $h(v,L,x)$ be the minimum length among all the lengths of walks starting in v and having maximum jump $a(v,L,x)$.

It is easy to see that the next proposition proved in [13] for finite graph remains valid in K (actually in every infinite graph finitely locally connected).

Proposition 1. *Let x be a Q-state such that $Es(F(x))=Es(x)$. Then for all $L\geqslant 1$ we have.*

(1) *If $a(v,L,x)\leqslant 0$ then $F_v^l(x)=x_v$, $\forall l\in\{1,\dots,L\}$.*

(2) *If $a(v,L,x)>0$ then $F_v^l(x)=s^{a(v,L,x)}(x_v)$, $\forall l\in\{h(v,L,x),\dots,L\}$.*

Let x be a Q-state with $Es(F^i(x))=Es(x)$, *foralli*. Let $G=(V,E)$ be a finite subgraph of K containing all the elements in supp(x) and such that for all $u\notin V$ all the edges incident with u belong to $E(x)$. Then the values $a(v,L,x)$ and $h(v,L,x)$ on K can be computed by considering only walks inside G. From Proposition 1, we know that the evolution of any vertex is completely determined by the values $a(v,L,x)$ and $h(v,L,x)$. Then, in order to know the evolution of a vertex $u\in V$ under the function F defined on K it is enough to consider its evolution under F defined on G.

Let P be a walk in $Es(x)$. The efficiency $e(P,x)$ of P on x is the quotient between its jump and its length. Since $|J(P,x)|\leqslant L(P)$ we have $|e(P,x)|\leqslant 1$.

Intuitively, the efficiency of a walk measures the potential that the walk has to impose the states of its vertices to its initial vertex. By instance, if two walks P_1 and P_2 starting at the same vertex v have both jump $a(v,L,x)$ but $L(P_1)<L(P_2)$ then P_1 (the most efficient) would impose its states to its initial vertex.

For a finite graph $G=(V,E)$ and a Q-state x on G such that $Es(F^i(x))=Es(x)$, $\forall i$ it is clear that for every $u\in V$ the sequence of states $(F_u^t(x))_{t\geqslant 0}$ is eventually periodic. Moreover, it is known that its period divides the lcm of the lengths of cycles with maximum efficiency in G (see [13]). Let $T(x)$ be the period of the NCN rule on K with initial condition x defined as the supremum over all the periods of vertices of K. From previous analysis we know that $T(x)$ is finite. Moreover, it divides the lcm of the lengths of cycles with maximum efficiency in $Es(F^{t_0}(x))$, where $t_0:=4|\text{supp}(x)|$ since $Es(F^{t_0+1}(x))=Es(F^{t_0}(x))$. Let e be the maximum of the efficiencies of cycles in $Es(F^{t_0}(x))$. We have proved the following result.

Theorem 2. *For every Q-state x the period of the NCN rule on K is finite. Moreover, it divides the lcm of the lengths of cycles in $Es(F^{t_0}(x))$ with efficiency e.*

The upper bound for the period of NCN rule in finite graphs has proved to be asymptotically optimal: there is an infinite family of finite graphs $(G_n)_{n\geqslant 1}$ such that for each n there is a Q-state y^n on G_n such that the period of the NCN rule *is* the lcm of the lengths of cycles with maximum efficiency on y^n. The graph G_n consists of n cycles $C_k,\ k=1,\dots,n$, joined by a central node v_0. The length of the cycle C_k is $4k$. The Q-state y^n on C_k is $0012(0112)^{k-1}$, starting in the vertex of C_k which is the neighbor of the central vertex v_0. Therefore, all cycles have efficiency $\frac{3}{4}$ and it is the maximum efficiency on G. Finally, the state of v_0 is set to 2.

Since every cycle evolves independently, the sequence of states in the central vertex will be $0012(0112)^{u-1}$, where $u=\text{lcm}\{4k:k=1,\dots,n\}$ (see Fig. 1).

We shall provide a construction of Q-states y^n on K, for $n\geqslant 1$ such that the period of the NCN rule on K is the lcm of the lengths of cycles with maximum efficiency on y^n. The Q-state y^n will be an embedding on K of the graph G_n previously described. It is not hard to see that if the period of the NCN is greater than 1 all the vertices in K will have period greater than 1. This implies that the stable skeleton is connected even if it is a proper subgraph of K. Therefore, the main problem we have to solve in order to construct the Q-state y^n on K is the interaction between the cycles due to the connectness of the stable skeleton. To get inside of the problem, we devote the next

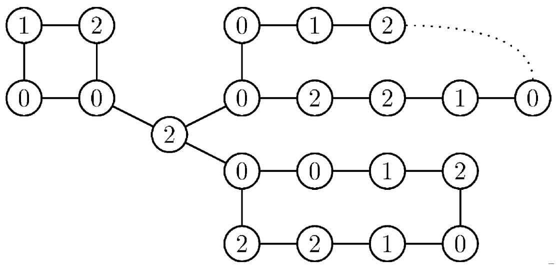

Fig. 1. NCN that reaches the bound $T(x) = e^{\Omega(\sqrt{|V|})}$ (see [13].)

section to the study of fully connected skeleton and we postpone the construction of y^n to Section 4.

3. Complete skeletons

In this section, we study the dynamic of NCN rule when the skeleton is the whole graph K. For $q \geqslant 5$ we prove that the period at any vertex is 1, i.e., after a finite time the state of the vertex remains constant. For $q = 4$ either the period of all vertices is 1 or the period of all vertices is four. We also prove that when $q = 3$ the dynamic is richer: for every even integer $t \geqslant 14$ there exists an Q-state on K with at most t states different from zero such that the period of every vertex is exactly t.

The basic units in the graph K are cycles of length four called *tiles*. It is clear that a tile in a skeleton can only have efficiency 0, $\frac{3}{4}$ or 1. Using a decomposition property we will see that a similar property holds for every cycle.

For two cycles γ_1 and γ_2 whose intersection is a path $P = i_k, \ldots, i_l$ we define their *union* $\gamma_1 \sqcup \gamma_2$ as the cycle $i_k \gamma_1 i_l \gamma_2 i_k$ containing no inner vertex of P and no inner edge of P.

We have the following lemma.

Lemma 3. *Let x be an Q-state over G. Let $\gamma_1 = i_0 i_1, \ldots, i_n$ and $\gamma_2 = j_0 j_1, \ldots, j_{n'}$ be two cycles such that $\gamma_1 \cap \gamma_2$ is a path P over K. Given a direction for $\gamma := \gamma_1 \sqcup \gamma_2$ and the induced directions over γ_1 and γ_2 we have*

$$J(\gamma, x) = J(\gamma_1, x) + J(\gamma_2, x).$$

Proof. The jump of γ is the addition of the jump of $\gamma_1 \setminus \mathring{P}$ and the jump of $\gamma_2 \setminus \mathring{P}$, where $\mathring{P}$ denotes the set of inner vertices of P. Since the path P is traversed in opposite direction by γ_1 and γ_2, the jump of P in γ_1 (denoted J_1) is minus the jump of P in γ_2 (J_2). Therefore, the jump of $\gamma_1 = \gamma_1 \setminus \mathring{P} + J_1$ and $\gamma_2 = \gamma_2 \setminus \mathring{P} + J_2$. Finally, the jump of γ is equal to the jump of γ_1 plus the jump γ_2. □

It is easy to see that the following property holds.

Corollary 4. *If γ_1 and γ_2 are two cycles of efficiencies* 0 *then the cycle* (*if ever*) $\gamma_1 \uplus \gamma_2$ *has efficiency* 0.

Previous properties are true for every cycle, in particular for tiles.

Lemma 5. *Let $q \geqslant 3$ be an integer and let x be a Q-state over K. If every tile has jump zero then the efficiency of the system $e(x)$ is zero.*

Proof. We prove that the jump of every cycle in K is zero. Let γ be a cycle in K. We prove the property by induction on the number n of inner points $(i+\frac{1}{2}, j+\frac{1}{2})$, $i,j \in \mathbb{Z}$ in the bounded region C of $\mathbb{R}^2 \backslash \gamma$. The case $n=1$ is when γ is a tile, by hypothesis its jump is zero.

Let us assume that the properties hold for every cycle with less than n inner points. Let v be any vertex in γ such that at least one incident edge of v in K has a part in the region is C. We cut the region C along a straight line starting in v and ending in a vertex of γ and having no other vertex in common with γ. This straight line defines a path in K. With this path and the cycle γ we build two cycles γ' and γ'' with less inner points than γ such that $\gamma' \uplus \gamma'' = \gamma$. By induction hypothesis we know that the jump of γ' and γ'' is zero. Therefore, from the Lemma 3 we conclude that the jump of γ is zero. □

Lemma 6. *Let q be an integer. Let x be a Q-state over K such that $Es(x)=K$.*
(1) *If $q \geqslant 5$ then $e(x)=0$.*
(2) *If $q=4$ then $e(x) \in \{0,1\}$.*
(3) *If $q=3$ then $e(x)=0$ or $e(x) \geqslant \frac{3}{4}$.*

Proof. Let α be the maximum jump over all the tiles. If $\alpha=0$ then from previous lemma we conclude. Since for $q \geqslant 5$ all tiles have jump zero, the only cases where $\alpha \neq 0$ are $q=3,4$.
(1) Case $q=4$. If $\alpha>0$ then at least one tile T has jump 4. Therefore the tile T has efficiency 1 which is obviously the maximum efficiency of the system.
(2) Case $q=3$. If $\alpha>0$ then at least one tile T has jump 3. Hence the tile T has efficiency $\frac{3}{4}$. Therefore, the maximum efficiency will be greater or equal to $\frac{3}{4}$. □

In order to get richer dynamics we can either consider complete skeleton and $q=3$ or to allow skeletons to be proper subgraph of K. Since we want to know the result for arbitrary q we have performed our constructions in the second framework. Nevertheless, we have proved that at least in term of the efficiency the case $q=3$ is rich enough as stated in the following theorem.

Theorem 7. *Let q be an integer.*
(1) *If $q \geqslant 5$ then the period of any vertex is* 1 *for all Q-state x with $Es(x)=K$.*
(2) *If $q=4$ then the period of any vertex is* 1 *or* 4 *for all Q-state x with $Es(x)=K$.*
(3) *If $q=3$, for any even integer $t \geqslant 14$ there exists an Q-state x with $Es(x)=K$ such that the period of any vertex is t. Moreover, this Q-state has at most t vertices with state different from zero.*

Proof. The two first cases are directly from the previous lemma. For $q=3$ we construct a rectangular cycle R with sides of lengths $\lfloor t/4 \rfloor$ and $\lceil t/4 \rceil$ with the following sequence over his vertices: i times 012 and j times 0112 for any $i, j > 0$ such that $3i + 4j = t$. We will prove that this is the unique cycle of maximum efficiency in this Q-state, and how the skeleton is complete, its period t will be the period of any vertex.

Indeed, a cycle sharing no edge with R has jump 0. Each cycle S sharing two or more paths with R can be replaced by a shorter cycle S' which shares only one path with R and with a bigger jump. So, the cycle S can be decomposed in two paths S_R and S_I such that S_R is a path inside R and S_I is a path outside R. Clearly, the subpath of S outside R is a shortest path between its ends. It can be seen that if the ends of the path S_I are not in opposite sides of R then the efficiency of S is either smaller than $\frac{1}{2}$ or smaller than the efficiency of R (smaller jump and more or equal length). If S_I is neither vertical nor horizontal but with both ends in opposite sides of R then S can be replaced by a cycle with the same length and bigger jump and with the required structure. Finally, if $L(S_R) < \frac{1}{2}t$ then $e(S) \leqslant \frac{2}{3} < e(R)$, and so, we only have to prove the result for a path S of length bigger than $\frac{3}{4}t$.

The jump of S is the jump of S_R plus the jump of S_I. Then $J(S) \leqslant L(S_R) + 2$. The length of S is either $L(S_R) + \lfloor t/4 \rfloor$ or $L(S_R) + \lceil t/4 \rceil$, so

$$e(S) \leqslant \frac{L(S_R)+2}{L(S_R)+\lceil t/4 \rceil} \leqslant 1 - \frac{\lceil t/4 \rceil - 2}{3\lceil t/4 \rceil} \leqslant \frac{2}{3} + \frac{8}{3t} < \frac{3}{4} \leqslant e(R) \qquad \square$$

Corollary 8. *For $q=3$ and for any $r \in \mathbb{Q} \cap [\frac{3}{4}, 1]$ exists a Q-state x with $Es(x) = K$ such that $e(x) = r$*

Proof. Let $r = a/b$ with a, b even integers. Has before, we construct a square cycle P with $j = 3K(q-p)$ times 0112 and $i = K(4p-3q)$ times 012 for any even K. The jump of P is $J(P,x) = 3K(3(q-p) + (4p-3q)) = 3Kp$ and its length is $L(P) = 3K(4p - 3q) + 4K(3(q-p)) = 3Kq$, so his efficiency is r and is a maximum efficiency cycle. $\square$

4. Our main theorem

In this section, we show how to construct a family $\{y^n : n \geqslant 1\}$ of Q-state on K each having a polynomial number t_n of non-quiescent states and period $T(y^n) = e^{\Omega(\sqrt{t_n})}$.

The skeleton of y^n contains n cycles C_k, for $k = 1, \ldots, n$, located in disjoint parts of K and such that their evolutions are independents. There is also a vertex v_0 which has roughly the same distance in the skeleton of y^n to each cycle C_k, for $k = 1, \ldots, n$. The evolution of v_0 will depend upon all the evolutions of the cycles C_k, $k = 1, \ldots, n$, in a manner similar to those shown in Fig. 1.

The construction of y^n is carried out as follows. Firstly, we construct, for each $k = 1, \ldots, n$, a Q-state x in a finite rectangular grid R_k and show that the skeleton of R_k on x is stable. The cycle C_k will be the one on the boundary of R_k. Secondly, we show how to embed x to a Q-state z^k of K such that the evolution of each vertex of

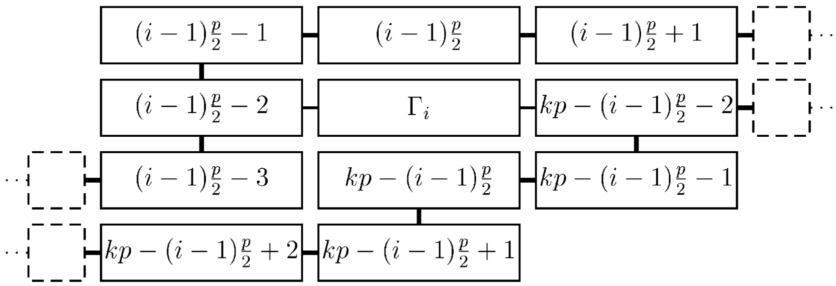

Fig. 2. Numbering of the cycle C_k and inner vertices.

K is the same as those appearing in the vertices of C_k. Thirdly, we construct y^n by merging all the Q-states z^k, for $k=1,\dots,n$ appropriately.

4.1. Construction of R_k

The construction of Rk and the Q-state x on R_k is based on the preliminary construction of a graph G_k and a Q-state w on G_k.

Let q be an even integer $q \geqslant 4$ and let p be an even integer such that $q \leqslant p \leqslant \lfloor \frac{3}{2} q \rfloor$.

Let B_i be the rectangular grid of size $i \times 2$ of $\mathbb{Z}^2$ given by

$$B_i = \{(s,t) : s = 0,\dots,i-1;\; t = 0,1\}.$$

Let $v \in \mathbb{Z}^2$ and $B \subseteq \mathbb{Z}^2$. We define

$$B + v = \{u + v : u \in B\},$$

where addition is taken in $\mathbb{Z}^2$.

The graph G_k is constructed iteratively as follows. Let $G_1 := B_{p/2}$. The graph G_i is the graph induced by $(V(G_{i-1}) \setminus \{u_i\}) \cup (B_{p/2+1} + t_i)$, where $u_i = (i-1)(p/2-3,2) + (2,-2)$, $t_i = u_i + (-2,2)$ and $V(G_{i-1})$ denotes the set of vertices of G_{i-1}. Then G_k can be decomposed in a cycle $C_k = v_0,\dots,v_{kp-1}, v_{kp} = v_0$ which clockwise follows the boundary of G_k and the set of vertices $\Gamma_i := t_i + (1,0)$, $i = 2,\dots,k$. The vertex Γ_i is called an inner vertex of G_k, for $i = 2,\dots,k$ (see Fig. 2). Let us denote by S_k the spanning subgraph of G_k whose set of edges is the set of all edges of C_k and all horizontal edges incident with inner vertices of G_k.

In the sequel all the additions on indices are taken modulo kp. Let $\mathscr{A}_k$ be the set of all sequences $a = (a_i : i = 0,\dots,kp-1)$ satisfying the following properties.

(1) For every $i = 0,\dots,kp-1$, $a_{i+1} \in \{a_i, s(a_i)\}$.
(2) For every $i = 0,\dots,kp/2-1$ the states a_i and a_{kp-1-i} are not adjacent in Q.
(3) For every $j = 0,\dots,k-1$ and for every $i = 0,\dots,kp-1$
  (a) $a_{i+jp} \in \{s^{-1}(a_i), a_i, s(a_i)\}$.
  (b) $a_{i+jp-3} \in \{s^{-2}(a_i), s^{-3}(a_i)\}$.
  (c) $a_{i+jp+3} \in \{s^{2}(a_i), s^{3}(a_i)\}$

Notice that the cyclic rotation function $\sigma : \mathscr{A}_k \to \mathscr{A}_k$ given by $\sigma_i(a) = a_{i+1}$ is an isomorphism on $\mathscr{A}_k$.

Let a be in $\mathscr{A}_k$. We define a Q-state $w(a)$ in G_k as follows. In C_k it assigns the state a_i to the vertex v_i, for $i = 0, \ldots, kp-1$. Let $i \in \{2, \ldots, k\}$ and let α_v denote the state of the vertex $\Gamma_i + v$, for $v \in \mathbb{Z}^2$ such that $\Gamma_i + v \in G_k$. Using Property 3a in the definition of $\mathscr{A}_k$ it can be seen that $\alpha_{-(1,1)}$ is adjacent with $\alpha_{(2,0)}$ in Q. Hence, we assign to the vertex Γ_i the state $\alpha_{-(1,1)}$ if $\alpha_{-(1,1)} = s(\alpha_{(2,0)})$ or the state $\alpha_{(2,0)}$ otherwise.

Lemma 9. $\forall k \geqslant 1$, *$Es(G_k, w(a)) = S_k$, for every $a \in \mathscr{A}_k$.*

Proof. From our assignment of states to the inner vertices of G_k and since $a_{i+1} \in \{a_i, s(a_i)\}$, for $i = 0, \ldots, kp-1$, we have that the subgraph S_k is contained in the skeleton of G_k.

Using the Property 2 in the definition of $\mathscr{A}_k$ we deduce that the set of all vertical edges in the skeleton of $w(a)$ on G_k is the set formed by all vertical edges in C_k.

Using the Property 3c we deduce that the horizontal edges of the skeleton not adjacent to any inner vertex are exactly the horizontal edges of C_k.

It remains to show that Γ_i is adjacent in the skeleton with neither $\Gamma_i + (0,1)$ nor $\Gamma_i + (0,-1)$.

From Property 3c we know that $\alpha_{(0,1)} \in \{s^2(\alpha_{-(1,1)}), s^3(\alpha_{-(1,1)})\}$ and from Property 3b that $\alpha_{(2,0)} \in \{s^{-2}(\alpha_{(0,1)}), s^{-3}(\alpha_{(0,1)})\}$. Let us assume that $\alpha_{(0,1)} = s^2(\alpha_{-(1,1)})$. Then $\alpha_{(2,0)} \in \{\alpha_{-(1,1)}, s^{-1}(\alpha_{-(1,1)})\}$. From the definition of the state of Γ_i we deduce that $\alpha_{(0,0)} = \alpha_{-(1,1)}$. From Property 3c we finally get $\alpha_{-(0,1)} \in \{s^2(\alpha_{-(1,1)}), s^3(\alpha_{-(1,1)})\}$. Therefore, the state $\alpha_{(0,0)}$ is adjacent with neither $\alpha_{-(0,1)}$ nor $\alpha_{(0,1)}$. The case when $\alpha_{(0,1)} = s^3(\alpha_{-(1,1)})$ can be handle in a similar way. □

Theorem 10. *For every $k \geqslant 1$ and for every $a \in \mathscr{A}_k$ we have that*

$$F(w(a)) = w(\sigma(a)).$$

Proof. Let v be a vertex in the cycle C_k not adjacent to any inner vertex in the skeleton S_k. Then $F_v(w(a)) = w_u(a)$ where u is the next clockwise neighbor of v. Then $w_u(a) = w_v(\sigma(a))$.

We prove the property for a vertex v in C_k having some inner vertex as neighbor in S_k. In this situation we know that the vertex v is either $\Gamma_i - (1,0)$ or $\Gamma_i + (1,0)$, for some $i = 2, \ldots, k$. Let us assume that $v = \Gamma_i - (1,0)$. From Property 1 we know that $\alpha_{(-1,1)} \in \{s(\alpha_{-(1,1)}), s^2(\alpha_{-(1,1)})\}$. Hence, if $\alpha_{(0,0)} = s(\alpha_{-(1,1)})$ then $F_{\Gamma_i-(1,0)}(w(a)) = w_{\Gamma_i+(-1,1)}(a) = w_{\Gamma_i-(1,0)}(\sigma(a))$. In the case that $\alpha_{(0,0)} = \alpha_{-(1,1)}$ we know from Property 1 that $\alpha_{(-1,0)} \in \{\alpha_{(0,0)}, s(\alpha_{(0,0)})\}$ and then $F_{\Gamma_i-(1,0)}(w(a)) = w_{\Gamma_i+(-1,1)}(a) = w_{\Gamma_i-(1,0)}(\sigma(a))$. The property for the vertex $\Gamma_i + (1,0)$ can be proved similarly.

Finally, doing a case analysis it is possible to show that $F_{\Gamma_i}(w(a)) = \beta_{(-1,1)}$ if $\beta_{(-1,1)} = s(\beta_{(2,0)})$ and that it is $\beta_{(2,0)}$ otherwise, where β_v denotes the state of the vertex $\Gamma_i + v$ in $F(w(a))$. Then $F_{\Gamma_i}(w(a))$ is the state associated to Γ_i by $w(\sigma(a))$. □

Corollary 11. *For every* $k \geqslant 1$ *and for every* $a \in \mathscr{A}_k$, $Es(G_k, F(w(a))) = S_k$.

Previous Corollary says that the skeleton of $w(a)$ on G_k is stable. It is not hard to see that the maximum efficiency of $w(a)$ is p/q. Moreover, the cycle C_k is the unique cycle with this efficiency.

Let R_k be the smallest rectangular grid on $\mathbb{Z}^2$ containing G_k. Then R_k is a $k(p/2-2)+2 \times 2k$ rectangular grid. We now describe how to obtain the Q-state $x(a)$ on R_k from the Q-state $w(a)$ on G_k. Let $v \in V(R_k) \backslash V(G_k)$ be with exactly two neighbors u and w in $V(G_k)$ and such that there exists a vertex v' in G_k adjacent to u and w. We assign to v the state of v'. We obtain $x(a)$ by iterating this process until a state be given to all vertices of R_k. Clearly, the maximum efficiency of $x(a)$ on R_k is p/q and $F(x(a)) = x(\sigma(a))$. Moreover, the sequence of states of $x(a)$ appearing in the boundary of R_k when we follow it clockwise is $\sigma^j(a)$, for some $j = 0, \ldots, kp-1$.

4.2. Construction of z^k

The Q-state $x(a)$ on R_k has a canonical extension to K (still denoted $x(a)$) which assigns the state 0 to all the vertices outside R_k (seen as a subgraph of K). It is not hard to see that the skeleton of $x(a)$ on K is not stable. Even worth if we iterate $x(a)$ with the NCN rule F the skeleton will increases and a cycle of greater efficiency than p/q will appear.

In order to obtain $z^k = z(a)$ we proceed to modify a finite number of states of $x(a)$ outside R_k. A vertex v in the boundary of R_k is called a *starting* vertex if its state is 1 and it has a neighbor in the boundary of R_k in the state 0. Let us denote by $v'_0, \ldots, v'_{kp-1}$ be the vertices appearing in the boundary of R_k when we follow it counter-clockwise. Then $v'_i = v_{kp-1-i}$, for all $i = 0, \ldots, kp-1$. Let b_i be the state of the vertex v'_i in $x(a)$. We associate to the vertex $v := v'_i$ the sequence b^v given by $b^v_1 = b_{i+1}, b^v_2 = b_{i+2}, \ldots, b^v_t = b_{i+t}$, where v'_{i+t} is a starting vertex and no vertex $v'_{i+1}, \ldots, v'_{i+t-1}$ is a starting vertex (still all additions are taken modulo kp). We denote by $l(u)$ the length of the sequence b^u. We associate the empty sequence to all the remaining vertices (those not in the boundary of R_k). The following procedure will modify the sequence b^v in some vertices of K. Let S be the set of all starting vertices.

Associating-sequences

begin
$W := S$;
while $(W \neq \emptyset)$ **do**.
  Let $v := v'_i \in W$;
  $u := v'_{i-1}$;
  $W := W \backslash \{v\}$.
  if b^v_2 and $x(a)_u$ are not neighbors in Q
  then
    $b^u := b^v_2, \ldots, b^v_{l(v)}$.

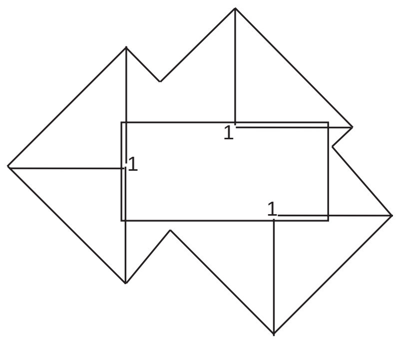

Fig. 3. Shape of the set $D(z(a))$. The inner rectangle represents the graph G_k.

$$W := W \cup \{u\}.$$

else

$$b^u := b^v_4, b^v_3, b^v_4, b^v_5, \ldots, b^v_{l(v)}.$$

endwhile
end

Let $(\bar{b^u})_{u\in K}$ be sequences obtained in all the vertices after applying the previous algorithm. Notice that only finitely many vertices have a non-empty sequence $\bar{b^u}$.

We say that we *complete* a vertex u in the direction d (with the sequence $\bar{b^u}$) if for all $i = 1, \ldots, l(u)$ we assign the state $\bar{b^u_i}$ to the vertex $u + id$. Let v be a vertex in the boundary of R_k. A direction d is *outgoing* for v if $v + d$ is a neighbor of v (in K) not in R_k. In order to obtain the Q-state $z(a)$, firstly, we *complete* all the vertices in their outgoing directions. Secondly, for each corner v of R_k with outgoing directions d_1 and d_2 we assign the state $\bar{b^v_i}$ to all the vertices (outside R_k) $v + i_1 d_1 + i_2 d_2$ with $i_1, i_2 \in \mathbb{Z},\ i_1, i_2 \geqslant 0$ and $i_1 + i_2 = i\ \forall i = 1, \ldots, l(v)$.

For a Q-state x we denote by $C(x)$ the vertex set of the (only) infinite connected component of the subgraph induced in K by $\mathbb{Z}^2 \backslash \mathrm{supp}(x)$ and let us denote by $D(x) := \mathbb{Z}^2 \backslash C(x)$.

From the construction of $z(a)$ it is clear that $\forall v \in D(z(a)),\ F_v(z(a)) = z_v(\sigma(a))$ (see Fig. 3)

From Proposition 1, we know that the evolution of every $v \notin D(z(a))$ is completely determined by the states in $D(z(a))$. More precisely we have.

Corollary 12. *Let $k \geqslant 1$ and $a \in \mathbb{A}_k$ be. Then*

- $F_v(z(a)) = z_v(\sigma(a))$ *for all* $v \in D(z(a))$.
- $F^t_v(z(a)) = 0$, *for all* $v \notin D(z(a))$ *and* $t = 0, \ldots, d-1$.
- $F^t_v(z(a)) = F^{t-d}_u(z(a))$, *for all* $v \notin D(z(a))$ *and* $t \geqslant d$.

where d is the (smallest) distance from v to $D(z(a))$ in K and u is any vertex in $D(z(a))$ to distance d of v.

4.3. Final result

In order to prove our main theorem let n be any integer $n \geqslant 1$. Let $k_1, \ldots, k_n$ be n integers and let $a^1, a^2, \ldots, a^n$ be n sequences such that for all $i = 1, \ldots, n$ the sequence a^i belongs to $\mathscr{A}_{k_i}$.

The addition of two Q-states x and y on K with disjoint supports is the Q-state z defined by $z_v = x_v$ if $y_v = 0$ and $z_v = y_v$ otherwise. This definition can be extended to n Q-states with disjoint supports in the obvious manner.

Let x be a Q state on K and $u \in \mathbb{Z}^2$. We define the shift of x on u as the Q-state $M^u(x)$ given by $M^u_v(x) = x_{u+v}$, for all $v \in \mathbb{Z}^2$. Then $M^0(x) = x$.

Let d be large enough such that there are $u_1, \ldots, u_n$ with $\bar{z}^i := M^{u_i}(z(a^i))$, for $i = 1, \ldots, n$ satisfy the following properties.

- The Q-states $\bar{z}^i$, for $i = 1, \ldots, n$ have disjoint support.
- The distance from $(0,0)$ to $D(\bar{z}^i)$ is d, for all $i = 1, \ldots, n$.

Let y^n be the Q-state obtained as the addition of all the Q-states $\bar{z}^i$, $i = 1, \ldots, n$. Clearly, the efficiency of y^n is q/p since each $\bar{z}^i$ has this efficiency, for $i = 1, \ldots, n$. Moreover, its skeleton is stable.

Theorem 13. *For every $q \geqslant 4$ and every even integer p such that $q \leqslant p \leqslant \lfloor \frac{3}{2} q \rfloor$ and, for every integer $n \geqslant 1$ and integers $k_1, k_2, \ldots, k_n$ there exists a Q-state y^n of K such that $T_{(0,0)}(y^n) = p \cdot lcm_{i=1,\ldots,n}\{k_i\}$.*

Proof. Given n and $k_1, \ldots, k_n$ let $a^1, a^2, \ldots, a^n$ be n sequences such that for all $i = 1, \ldots, n$ the sequence a^i belongs to $\mathscr{A}_{k_i}$.

We define y^n as described above. Since the skeleton of y^n is stable we know from Proposition 1 that the evolution of the vertex $(0,0)$ depends only on the evolution of a vertex $v_i \in D(z(a^i))$ with distance d to it.

Our goal now is to choose appropriated sequences a^i, $i = 1, \ldots, n$ so as the period in vertex $(0,0)$ is $p \cdot \mathrm{lcm}_{i=1,\ldots,n}\{k_i\}$.

Let a' be any sequence in $\mathscr{A}_1$ that starts as follows $(1, 2, 2, 3, 4, \ldots, q-1, 0)$ and let a'' be the sequence obtained from a' by the following modification $(1, 1, 2, 3, \ldots, q-1, 0)$. The sequences a^i is composed by $k_i - 1$ times the subsequences a' followed by the subsequence a'', that is

$$a^i = (1, 2, 2, 3, \ldots, 0)^{(k_i - 1)}(1, 1, 2, 3, \ldots, 0)$$

Then the sequence of states of the vertex $(0,0)$ is given by

$$(1, 2, 2, \ldots, 0)^{\kappa - 1}(1, 1, 2, \ldots, 0)$$

where $\kappa = \mathrm{lcm}_{i=1,\ldots,m}\{k_i\}$. Therefore $T_v(y) = p \cdot \kappa$. $\square$

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