Decision Support

Linear models for stockpiling in open-pit mine production scheduling problems

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A R T I C L E   I N F O

A B S T R A C T

The open pit mine production scheduling (OPMPS) problem seeks to determine when, if ever, to extract each notional, three-dimensional block of ore and/or waste in a deposit and what to do with each, e.g., send it to a particular processing plant or to the waste dump. This scheduling model maximizes net present value subject to spatial precedence constraints, and resource capacities. Certain mines use stockpiles for blending different grades of extracted material, storing excess until processing capacity is available, or keeping low-grade ore for possible future processing. Common models assume that material in these stockpiles, or "buckets," is theoretically immediately mixed and becomes homogeneous.

We consider stockpiles as part of our open pit mine scheduling strategy, propose multiple models to solve the OPMPS problem, and compare the solution quality and tractability of these linear-integer and nonlinear-integer models. Numerical experiments show that our proposed models are tractable, and correspond to instances which can be solved in a few seconds up to a few minutes in contrast to previous nonlinear models that fail to solve.

1. Introduction

Open pit mine production scheduling (OPMPS) is a decision problem involving which blocks, within the final pit limits, should be mined in each year, and where the blocks should be sent, e.g., mill, waste dump or stockpile, to maximize the net present value (NPV) subject to the constraints that: (i) mining and processing consume limited resources and affect the production profile in each period; and (ii) spatial precedence must be obeyed among the blocks (Fig. 1).

In open pit mine scheduling, the question arises as to how mathematically to model the stockpile and determine a strategy, and how to assess the value associated with using a stockpile. While some researchers do not consider a stockpile as part of OPMPS, others suggest using a stockpile without providing the mathematical framework. In this research, we focus on proposing tractable models which provide practical solutions.

Initially, researchers proposed linear programs to solve OPMPS without considering a stockpile. Johnson (1969) describes the first such model to maximize net present value (NPV) of an open pit mine while determining whether each block should be sent to the mill or the waste dump, subject to precedence and operational resource constraints. Because his model contains only continuous-valued variables, his precedence constraints enforce that in order to extract a certain amount of block b', at least that same amount of predecessor block b must be extracted. The author uses Dantzig–Wolfe decomposition to solve several instances. Given hardware and software limitations at the time, he illustrates with some small examples.

An important challenge in solving OPMPS is that model instances can contain many blocks and time periods, and each block-time period combination has an associated binary decision variable in order to capture the more realistic constraint that all of a predecessor block must be extracted before any of a successor block is extracted. One way to decrease the number of decision variables in these linear-integer programs is to aggregate some blocks with similar characteristics. Askari-Nasab, Frimpong, and Szymanski (2007) discuss different aggregation techniques that can be used to fit the geology of the deposit and the time fidelity of the model. They also develop an open-pit production method which depicts the stochastic dynamic expansion of an open pit using discrete incremental pushbacks in different directions.

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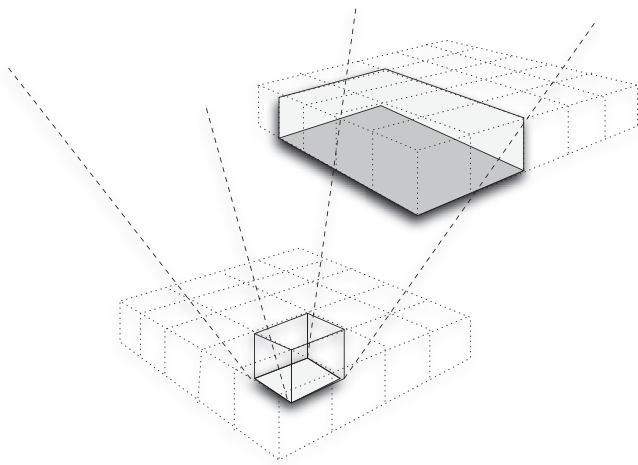
Fig. 1. Block precedence relationships. In order to extract the lower blocks, all upper blocks inside a predefined slope should be previously extracted. Figure courtesy of Nelson Morales.

Ramazan (2007) uses the concept of "fundamental trees" to aggregate blocks for an open pit production scheduling problem. Boland, Dumitrescu, Froyland, and Gleixner (2009) suggest that variables or constraints which are determined to be "similar" according to some criteria can be grouped together into new variables or constraints, called aggregates. The new OPMPS problem is then solved, causing some decisions to lose their fidelity in the aggregated model. By disaggregating, i.e., reverting to the original variables, a solution for the initial problem, which is usually not optimal and possibly infeasible, is obtained. Jélvez, Morales, Nancel-Penard, Peypouquet, and Reyes (2016) present a number of heuristics to tackle the open-pit block scheduling problem. Their approach is mainly based on block aggregation. The authors first solve the aggregated problem and then obtain a feasible solution for the original instance.

Bienstock and Zuckerberg (2010) provide a new algorithm for solving the linear programming relaxation of the precedence constrained production scheduling problem by reformulating it such that many constraints are modeled as a single one. They also consider multiple processing options. Their maximum weight closure problem can be solved as a minimum cut problem with a small number of side constraints, making it amenable to Lagrangian-based approaches. Chicoisne, Espinoza, Goycoolea, Moreno, and Rubio (2012) propose a new algorithm to solve linear programming relaxations of large instances of the same problem, and a set of heuristics to solve the corresponding integer program.

Martinez and Newman (2011) present a mixed-integer model to schedule long- and short-term underground production which minimizes deviations from preplanned production quantities while adhering to operational constraints. The authors develop an optimization-based decomposition heuristic that solves large instances quickly. O'Sullivan and Newman (2015) schedule extraction and backfill at an underground Lead–Zinc mine that uses three different underground methods; their heuristic enables them to solve real-world instances.

Shishvan and Sattarvand (2015) present a metaheuristic approximation based on Ant Colony Optimization for open-pit mine production planning which considers any type of objective function and nonlinear constraints. Montiel and Dimitrakopoulos (2015) propose a risk-based method which incorporates geological uncertainty to optimize mining operations comprised of multiple pits, stockpiles, blending requirements, processing paths, operating alternatives and transportation systems. Their method

perturbs an initial solution iteratively to improve the objective function. Lamghari and Dimitrakopoulos (2016) and, similarly, de Freitas Silva, Dimitrakopoulos, and Lamghari (2015) propose different heuristics such as tabu search and variable neighborhood descent to solve models that consider metal uncertainty and multiple destinations for the extracted material; low-grade material sent to the stockpile is mixed homogeneously, and the corresponding average grade is successively approximated.

Although linear and mixed integer programming models are recognized as having significant potential for optimizing production scheduling in both open pit and underground mines, most of these approaches focus on the extraction sequence and do not consider the material flow post-extraction. In particular, the use of stockpiling to manage processing plant capacity, and the interplay of material flows from the mine to a stockpile, the mine to a processing plant, and a stockpile to a plant, have not been treated as an integrated part of mine extraction sequence optimization. While industrial uses of mine planning software with stockpiling exist, these have limited benefit due to the nature of their modeling and solution techniques.

1.1. Existing industrial software

While some mining software such as Mintec (2013) and MineMax (2016) have tried to consider the stockpile as part of open pit mine scheduling, such software does not guarantee global optimal solutions. Whittle, one of the leading pieces of software in mine planning, has a stockpiling module and considers mixing material with different grades in the stockpile:

As material is moved to the stockpile, the tonnage and metal information is accumulated, so that at any point in time, the average grade is known. Stock withdrawals are considered to be at the average grade. Stockpiles are only used if they return a positive cash flow (Whittle, 2010).

Whittle does not use optimization techniques to model the stockpile, so there is no guarantee of obtaining an optimal solution with respect to the number of stockpiles and/or the grade contained in each stockpile. Academic researchers have been developing models to address these shortcomings.

1.2. Linear-integer models considering a stockpile

Smith (1999) uses mixed integer programming to solve a short-term production scheduling problem with blending, considering stockpiles both at the mine and at the mill. He notes that correctly capturing the contents of the stockpile requires nonlinear constructs, and enhances tractability of the original model by introducing piecewise linear constructs to approximate separable terms (after reformulation) representing the product of the average grade in the stockpile and the quantity retrieved from the stockpile in a given time period. After aggregation and variable elimination, he applies the model results to a phosphorus mine in Idaho. This research represents an early attempt to correctly model the grade of a stockpile, but requires approximations whose accuracies are not quantified, to ensure tractability.

Caccetta and Hill (2003) propose an exact approach to solve a monolithic OPMPS problem by defining variables representing whether a block is mined by time period *t*. The model includes constraints on: precedence, operational resources, and processing grade requirements. They also discuss the possibility of considering a stockpile in their model but without an associated mathematical formulation. The authors propose a branch-and-cut strategy combined with a heuristic. Asad (2005) describes a simple optimization model designed to assess the tradeoffs between cut-off grades and stockpile levels for a two-mineral deposit. His static

model omits production scheduling decisions. Ramazan and Dimitrakopoulos (2013) explain that the OPMPS problem typically contains uncertainty in the geological and economic input data. They use a stochastic framework to incorporate stockpiling since the amount of material to be stockpiled is determined by the block grades in the orebody model. In these models, the authors ignore mixing of material in the stockpile. Koushavand, Askari-Nasab, and Deutsch (2014) quantify oregrade uncertainty by including a term for its cost in the objective function; their model captures typical constraints on extraction and processing limits, and on block precedence, as well as on blending, and on over- and underproduction. Stockpile levels are bounded above and below, and are tracked in aggregate by time period; the authors demonstrate their model using a case study in which they assume that the stockpile has its grade set a priori and that it is used to mitigate uncertainty, i.e., overproduction can be carried over until the next time period. Smith and Wicks (2014) use a mixed-integer program (MIP) that maximizes recovered copper and accounts for constraints on shovel, extraction, stockpiling, and processing capacities, as well as blending. Here, the stockpiling constraints result in an optimistic bound on the model, in that each block is retrieved from the stockpile having preserved its characteristics upon entry to the stockpile. The authors' life-of-mine model, solved using a sliding time window heuristic to incorporate a 60-month horizon, yields information regarding stripping ratios and qualities and quantities of ore mined.

1.3. Nonlinear-integer models considering a stockpile

Nevertheless, some researchers do consider material mixing in the stockpile. When placing an ore block on a stockpile, the block characteristics (e.g., grade and tonnage) are known. However, as blocks are mixed in the stockpile, the characteristics of the material removed from the stockpile must be treated as variables. Since the amount of ore removed from the stockpile is not known a priori, the model has some non-convex, nonlinear constraints. Efforts to solve this problem result in local optimal solutions or consist of linearizing the model, which might introduce unrealistic assumptions.

Tabesh, Askari-Nasab, and Peroni (2015) acknowledge that stockpiling should theoretically be modeled nonlinearly to optimize a comprehensive open-pit mine plan, and linearizes the formulation by using a "sufficient number" of stockpiles, each with a tight range of grades. No numerical results are given, however. (We will return to this model later.)

Although there have been efforts to consider stockpiling as part of OPMPS, some of these models result in locally optimal solutions and/or are intractable for big data sets. Attempting to decrease the size of the problem instances results in aggregation, which causes a loss of information regarding each type of material (Tabesh & Askari-Nasab, 2011).

Bley, Boland, Froyland, and Zuckerberg (2012a) propose two different models considering *one* stockpile with the following assumptions:

1. Material in the stockpile mixes, resulting in a grade equal to the average grade of all the material inside the stockpile.
2. Material is extracted from the stockpile at the beginning of each period, so the grade of the resulting material is the average of that of the material at the end of the previous period.

In Section 3.2, we present $(\mathcal{P}^b)$, which tracks the ore and mineral in the stockpile in each period, considering material mixing by adding a non-convex quadratic constraint for each period. In Section 3.3, we discuss $(\mathcal{P}^w)$, in which the fraction of each block in the stockpile in each period is tracked, and additional non-convex constraints force the fraction of each block in the stockpile that is

sent to be processed in a given time period to be the same. Bley et al. (2012a) prove that $(\mathcal{P}^b)$ and $(\mathcal{P}^w)$ are equivalent, but the latter model provides a stronger formulation of the problem, resulting in a better upper bound.

Bley et al. (2012a) focus on exact algorithmic approaches. They study a relaxation of $(\mathcal{P}^w)$ by removing the non-linear constraints, and instead enforcing these restrictions using a scheme, integrated within a branch-and-bound framework, that (i) branches on the variable representing the value of the proportion of metal (versus ore) removed from the stockpile in each time period, and (ii) forces the violation of all non-linear constraints to be arbitrarily close to 0. Additionally, the authors propose a primal heuristic to obtain feasible solutions of the exact problem from a relaxed solution, and cuts and inequalities to strengthen the relaxation. Finally, they apply these techniques on two small instances, showing the impact of each solution procedure they propose.

Our research, by contrast, focuses on proposing new models, rather than on developing new algorithms, and compares how their assumptions affect solution quality and tractability. These linear-integer models include blending requirements without unrealistic assumptions, and yield good approximations using state-of-the-art methodologies on large-scale instances.

We organize the remainder of this paper as follows. In Section 2, we explain an existing model that does not incorporate stockpiling; in Section 3, we present existing nonlinear models that incorporate stockpiling. In Section 4, we propose linear models with stockpiling. In Section 5, we graphically represent the difference between our proposed models, and in Section 6, we compare the results. We conclude with Section 7.

2. Lower bound model

In this section, we present the formulation of a model that provides a lower bound on the objective function value of the OPMPS problem in which the option of stockpiling does not exist; such a model can be found in Caccetta and Hill (2003), Boland et al. (2009), and as a special case of Bienstock and Zuckerberg (2010). The first section introduces notation, and the following sections provide the math. We use the term "material" to include ore, i.e., rock that contains sufficient minerals including metals that can be economically extracted, and to include waste.

2.1. Notation

Indices and sets:

$b \in \mathcal{B}$: blocks; $1, \ldots, B$

$\hat{b} \in \hat{\mathcal{B}}_b$: blocks that must be mined directly before block b

$r \in \mathcal{R}$: resources $\{1 = \text{mine}, 2 = \text{mill}\}$

$t \in \mathcal{T}$: time periods; $1, \ldots, T$

Parameters:

δ_t : discount factor for time period t (fraction)

C^m : mining cost per ton of material (dollars per ton)

C^p : processing cost per ton of material (dollars per ton)

P : profit generated per ton of metal (dollars per ton)

W_b : tonnage of block b (ton)

M_b : metal obtained by completely processing block b (ton)

Decision variables:

y_{bt}^m : fraction of block b mined in time period t

y_{bt}^p : fraction of block b mined in time period t and sent (directly) to the mill

y_{bt}^w : fraction of block b mined in time period t and sent to waste

x_{bt} : 1 if block b has finished being mined by time t; 0 otherwise

2.2. Model without stockpiling ($\mathcal{P}^{ns}$)

The following model omits stockpiling:

$$(\mathcal{P}^{ns}) : \max \sum_{t \in \mathcal{T}} \delta_t \left[P\left(\sum_{b \in \mathcal{B}} M_b y_{bt}^p \right) - C^p \left(\sum_{b \in \mathcal{B}} W_b y_{bt}^p \right) \right.$$
$$\left. - C^m \left(\sum_{b \in \mathcal{B}} W_b y_{bt}^m \right) \right] \tag{1}$$

$$y_{bt}^p + y_{bt}^w = y_{bt}^m \quad \forall b \in \mathcal{B}, \forall t \in \mathcal{T} \tag{2}$$

$$\sum_{t \in \mathcal{T}} y_{bt}^m \leq 1 \quad \forall b \in \mathcal{B} \tag{3}$$

$$x_{bt} \leq \sum_{t' \leq t} y_{bt'}^m \quad \forall b \in \mathcal{B}, \forall t \in \mathcal{T} \tag{4}$$

$$\sum_{t' \leq t} y_{bt'}^m \leq x_{\hat{b}t} \quad \forall b \in \mathcal{B}, \hat{b} \in \hat{\mathcal{B}}_b, t \in \mathcal{T} \tag{5}$$

$$(x, y) \in \Omega \quad \text{(other constraints)} \tag{6}$$

The objective function is the sum of the revenues of blocks sent directly to the mill, minus the sum of the extraction and processing costs. All terms are multiplied by an appropriate discount rate according to the time period, t.

The first constraint forces the material sent to the mill or waste to equal the quantity of extracted material. Constraint (3) ensures that extracted fractions of each block summed across all time periods must be less than or equal to one. Constraint (4) forces the sum of the fractional variables to 1 by time t if the block has been mined by that time. Constraint (5) enforces mining precedence constraints by ensuring that for each block, all predecessors are completely mined before any amount of the successor block is mined. Constraint (6) might represent geometrical and operational restrictions (e.g., block-level or bench-phase scheduling, mining and processing bounds, blending constraints, and/or the maximum number of phases opened), and could involve bound and integrality constraints on x and y.

3. Nonlinear models that consider stockpiling

In this section, we provide nonlinear formulations that consider a stockpile. Because we propose models with just one stockpile, we define "buckets" that represent different parts of a stockpile, where each bucket incorporates material within a specific grade range. The grade of material when removing it from the stockpile is the minimum grade of the associated bucket. First, we define additional notation:

3.1. Notation

Indices and sets:

$k \in \mathcal{K}$: buckets; $1, \ldots, K$

Parameters :

C^h : rehandling cost per ton of material (dollars per ton)

L : average grade in the stockpile (grams per ton)

L_k : average grade in bucket k (grams per ton)

Decision variables:

y_{bt}^s : fraction of block b mined in time period t and sent to the stockpile

$\bar{y}_{bkt}^s$: fraction of block b mined in time period t and sent to bucket k of the stockpile

z_{bt}^p : fraction of block b sent from the stockpile to the mill in time period t

z_{bt}^s : fraction of block b remaining in the stockpile at the end of time period t

f_t : relative proportion of blocks from the stockpile processed in time period t

i_t^p, m_t^p : tonnage of ore and metal sent from the stockpile to the mill in time period t, respectively

i_t^s, m_t^s : tonnage of ore and metal remaining in the stockpile at the end of time period t, respectively

$\bar{i}_{kt}^p$: tonnage of ore sent from bucket k in the stockpile to the mill in time period t

$\bar{i}_{kt}^s$: tonnage of ore remaining in bucket k in the stockpile at the end of time period t

3.2. Basic model that considers stockpiling ($\mathcal{P}^b$)

The OPMPS+S with a single stockpile is similar to the formulation in Section 2.2 except that the objective function and constraint (2) are modified and five more constraints are added. Close variants of the following two formulations are given in Bley et al. (2012a). The objective function becomes:

$$(\mathcal{P}^b) : \max \sum_{t \in \mathcal{T}} \delta_t \left[P\left(\sum_{b \in \mathcal{B}} M_b y_{bt}^p + m_t^p \right) - C^p \left(\sum_{b \in \mathcal{B}} W_b y_{bt}^p + i_t^p \right) \right.$$
$$\left. - C^m \left(\sum_{b \in \mathcal{B}} W_b y_{bt}^m \right) - C^h i_t^p \right] \tag{7}$$

The profit of the blocks sent from the stockpile to the mill is added to ($\mathcal{P}^{ns}$)'s objective function, and processing and rehandling costs are subtracted. Constraint (2) becomes:

$$y_{bt}^p + y_{bt}^w + y_{bt}^s = y_{bt}^m \quad \forall b \in \mathcal{B}, \forall t \in \mathcal{T} \tag{8}$$

The original constraints from ($\mathcal{P}^{ns}$) are:

$$\sum_{t \in \mathcal{T}} y_{bt}^m \leq 1 \quad \forall b \in \mathcal{B} \tag{3 revisited}$$

$$x_{bt} \leq \sum_{t' \leq t} y_{bt'}^m \quad \forall b \in \mathcal{B}, \forall t \in \mathcal{T} \tag{4 revisited}$$

$$\sum_{t' \leq t} y_{bt'}^m \leq x_{\hat{b}t} \quad \forall b \in \mathcal{B}, \hat{b} \in \hat{\mathcal{B}}_b, t \in \mathcal{T} \tag{5 revisited}$$

$$(x, y) \in \Omega \quad \text{(other constraints)} \tag{6 revisited}$$

The five new constraints are:

$$i_t^p \leq i_{t-1}^s \quad \forall t \in \mathcal{T} \tag{9}$$

$$m_t^p \leq m_{t-1}^s \quad \forall t \in \mathcal{T} \tag{10}$$

$$i_t^s = \begin{cases} \sum_{b \in \mathcal{B}} W_b y_{bt}^s & t = 1 \\ i_{t-1}^s - i_t^p + \sum_{b \in \mathcal{B}} W_b y_{bt}^s & t \in \mathcal{T} : t \geq 2 \end{cases} \tag{11}$$

$$m_t^s = \begin{cases} \sum_{b \in \mathcal{B}} M_b y_{bt}^s & t = 1 \\ m_{t-1}^s - m_t^p + \sum_{b \in \mathcal{B}} M_b y_{bt}^s & t \in \mathcal{T} : t \geq 2 \end{cases} \quad (12)$$

$$\frac{m_t^p}{i_t^p} \leq \frac{m_{t-1}^s}{i_{t-1}^s} \quad \forall t \in \mathcal{T} \quad (13)$$

Constraints (9) and (10) ensure that what we send from the stockpile to the mill in time period t is less than or equal to the material and the metal, respectively, in the stockpile in time period $t - 1$. Constraints (11) and (12) enforce inventory balance for an initial time period and a general time period t, ensuring that the amount of material and metal, respectively, in the stockpile during time period t is equal to that of the last period plus anything that was added and minus anything sent to the mill from the stockpile. Constraint (13) forces the ratio of the metal contained in the material (i.e., grade) sent to the processing plant to be less than or equal to that ratio in the stockpile at the end of the previous time period.

3.3. Warehouse model ($\mathcal{P}^w$)

The formulation with one stockpile and homogeneous mixing requires different variable definitions but follows the same logic as the formulation of the Basic Model ($\mathcal{P}^b$). The new variables (defined in Section 3.1) express stockpile amounts in terms of fractions of the block instead of in terms of tons, which is necessary to track the grade of each block going to the stockpile. The new objective function is:

$$(\mathcal{P}^w) : \max \sum_{t \in \mathcal{T}} \delta_t \left[P \left(\sum_{b \in \mathcal{B}} M_b (y_{bt}^p + z_{bt}^p) \right) - C^p \left(\sum_{b \in \mathcal{B}} W_b (y_{bt}^p + z_{bt}^p) \right) \right.$$

$$\left. - C^m \left(\sum_{b \in \mathcal{B}} W_b y_{bt}^m \right) - C^h \left(\sum_{b \in \mathcal{B}} W_b z_{bt}^p \right) \right] \quad (14)$$

subject to:

$$y_{bt}^p + y_{bt}^w + y_{bt}^s = y_{bt}^m \quad \forall b \in \mathcal{B}, \forall t \in \mathcal{T} \quad (8 \text{ revisited})$$

$$\sum_{t \in \mathcal{T}} y_{bt}^m \leq 1 \quad \forall b \in \mathcal{B} \quad (3 \text{ revisited})$$

$$x_{bt} \leq \sum_{t' \leq t} y_{bt'}^m \quad \forall b \in \mathcal{B}, \forall t \in \mathcal{T} \quad (4 \text{ revisited})$$

$$\sum_{t' \leq t} y_{bt'}^m \leq x_{\hat{b}t} \quad \forall b \in \mathcal{B}, \hat{b} \in \hat{\mathcal{B}}_b, t \in \mathcal{T} \quad (5 \text{ revisited})$$

$$z_{bt}^s = \begin{cases} y_{bt}^s & t = 1 \\ z_{b,t-1}^s + y_{bt}^s - z_{bt}^p & t \in \mathcal{T} : t \geq 2 \end{cases} \quad \forall b \in \mathcal{B}, t \in \mathcal{T} \quad (15)$$

$$\frac{z_{bt}^p}{z_{bt}^p + z_{bt}^s} = f_t \quad \forall b \in \mathcal{B}, t \in \mathcal{T} \quad (16)$$

$$(x, y) \in \Omega \quad \text{(other constraints)} \quad (6 \text{ revisited})$$

As before, constraint (8) forces the material sent directly to the mill, to the stockpile, or to waste to be equal to the quantity of extracted material. The following constraints duplicate those in ($\mathcal{P}^{ns}$): Constraint (3) ensures that extracted fractions of each block summed across all time periods must be less than or equal to one. Constraint (4) forces the sum of the fractional variables to 1 by time t if the block has been mined by that time. Constraint (5) enforces mining precedence constraints by ensuring that

for each block, all predecessors are completely mined before any amount of the successor block is mined. Constraints specific to ($\mathcal{P}^w$) include: constraint (15) indicates that the fraction of block b remaining in the stockpile at period t will be the remaining fraction from the previous period, plus the fraction of b extracted at period t and sent to the stockpile, minus the fraction of b sent from the stockpile to mill at period t. Constraint (16), which introduces a new variable, f_t, requires that the relative proportion of each block in the stockpile that is processed is the same for all blocks in each time period. As in ($\mathcal{P}^{ns}$), constraint (6) might represent geometrical and operational restrictions (e.g., block-level or bench-phase scheduling, mining and processing bounds, blending constraints, and/or the maximum number of phases opened), and could involve bound and integrality constraints on x and y. Bley et al. (2012a) prove that ($\mathcal{P}^b$) and ($\mathcal{P}^w$) are equivalent.

4. Approximate linear models

In Sections 4.1–4.3, we formalize results from models in the literature, i.e., Akaike and Dagdelen (1999), Hoerger, Seymour, and Hoffman (1999) and Tabesh et al. (2015), respectively. In the latter case, the authors present a model that is similar to *K*-bucket (see Section 4.3), in which the authors categorize the possible grades in the buckets; there are, however, three differences: (i) they define a lower and an upper bound for the *average* grade sent to each bucket in each period, and (ii) they assess an "output grade" from each bucket, which does not necessarily correspond to the upper or to the lower bound, and (iii) there is no linking constraint between the buckets. The model we present in Section 4.4 is new. We conclude Section 4 with a summary of our models and an example of where they appear seminally in the literature, if at all.

4.1. Upper bound model ($\mathcal{P}^{ub}$)

A model that provides an upper bound on ($\mathcal{P}^w$) can be obtained by removing the nonlinear constraint (16) such that each block can be sent (individually, maintaining its original characteristics) from the stockpile to the processing plant. The solution of this upper bound model, ($\mathcal{P}^{ub}$), is generally infeasible for the basic and warehouse models. A different upper bound can be obtained by removing non-linear constraints (13) from ($\mathcal{P}^b$); however, Bley et al. (2012a) show that the corresponding upper bound is never better than that provided by ($\mathcal{P}^{ub}$).

4.2. L-bound model ($\mathcal{P}^{lb}$)

We can assume that the stockpile has a pre-defined output grade, denoted by *L*. In order to obtain a feasible solution for ($\mathcal{P}^b$) or ($\mathcal{P}^w$), only blocks with grade greater than or equal to *L* may be sent to the stockpile. Hence, we replace constraint (13) in ($\mathcal{P}^b$) by the following constraints:

$$m_t^p = L \cdot i_t^p \quad \forall t \in \mathcal{T} \quad (17)$$

$$y_{bt}^s = 0 \quad \forall b \in \mathcal{B} \text{ such that } \frac{M_b}{W_b} < L, \forall t \in \mathcal{T} \quad (18)$$

Lemma 1. *Let v^b be the optimal objective function value of the nonlinear model ($\mathcal{P}^b$), and v^{lb} be that of model ($\mathcal{P}^{lb}$); then $v^{lb} \leq v^b$.*

Proof. We will show that the optimal solution of ($\mathcal{P}^{lb}$) is feasible for ($\mathcal{P}^b$) to demonstrate that an optimal solution for the latter problem must be at least as good as that for the former. The objective functions of the two models are the same, and all constraints are the same with the exception of (13) (see Table 1). So, we only need to prove that a solution satisfying (17) and (18) satisfies (13):

Table 1

Summary of principal open-pit mine scheduling models that include inventory; we provide: (i) our naming convention for each model, (ii) the section of this paper in which we introduce it, (iii) the associated variables, objective function and constraints, and (iv) an example of related, seminal work that uses such a model or a close approximation thereof.

Model	Section	Variables	Objective	Constraints	Related seminal reference
$(\mathcal{P}^{ns})$	2.2	$x_{bt}, y_{bt}^m, y_{bt}^p, y_{bt}^w$	(1)	(2)–(6)	Johnson (1969)
$(\mathcal{P}^b)$	3.2	$x_{bt}, y_{bt}^p, y_{bt}^w, y_{bt}^s, y_{bt}^m$ $i_t^s, i_t^p, m_t^s, m_t^p$	(7)	(3)–(6), (8)–(13)	Bley et al. (2012a)
$(\mathcal{P}^w)$	3.3	$x_{bt}, y_{bt}^m, y_{bt}^p, y_{bt}^w$ $y_{bt}^s, z_{bt}^p, z_{bt}^s, f_t$	(14)	(3)–(6), (8), (15), (16)	Bley et al. (2012a)
$(\mathcal{P}^{ub})$	4.1	$x_{bt}, y_{bt}^m, y_{bt}^p, y_{bt}^w$ $y_{bt}^s, z_{bt}^p, z_{bt}^s$	(14)	(3)–(6), (8), (15)	Akaike and Dagdelen (1999)
$(\mathcal{P}^{lb})$	4.2	$x_{bt}, y_{bt}^m, y_{bt}^p, y_{bt}^w, y_{bt}^s$ $i_t^p, m_t^p, i_t^s, m_t^s$	(7)	(3)–(6), (8)–(12), (17), (18)	Hoerger et al. (1999)
$(\mathcal{P}^{kb})$	4.3	$x_{bt}, y_{bt}^m, y_{bt}^p, y_{bt}^w$ $\bar{y}_{bkt}^s, \bar{i}_{kt}^p, m_t^p, \bar{i}_{kt}^s, m_t^s$	(7)	(3)–(6), (10), (12), (21)–(27)	Hoerger et al. (1999) Tabesh et al. (2015)
$(\mathcal{P}^{la})$	4.4	$x_{bt}, y_{bt}^m, y_{bt}^p, y_{bt}^w, y_{bt}^s$ $i_t^p, m_t^p, i_t^s, m_t^s$	(7)	(3)–(6), (8)–(12), (28), (29)	New model

Constraints (11) and (12), which are contained in $(\mathcal{P}^{lb})$, can be rewritten as:

$$i_t^s = \sum_{t' \le t}\sum_{b \in \mathcal{B}} W_b y_{bt'}^s - \sum_{t' < t} i_{t'}^p \quad t \in \mathcal{T} \tag{19}$$

$$m_t^s = \sum_{t' \le t}\sum_{b \in \mathcal{B}} M_b y_{bt'}^s - \sum_{t' < t} m_{t'}^p \quad t \in \mathcal{T} \tag{20}$$

by cumulating on t. (Note that the case in which $t = 1$ is addressed by the loose inequality for the summation on time in the first term and the corresponding strict inequality in the second for both expressions: i_t^s and m_t^s.)

By the contrapositive, constraint (18) implies that if $y_{bt}^s > 0$ then $M_b \ge L \cdot W_b$. If we multiply Eq. (19) by L and compare it to Eq. (20), we see from (17) that the second term on the right hand side of both equations is the same, while the first term is larger in (20), because the condition stated in (18) does not hold. This implies that the left hand side of (20) is larger than that of (19) (with the left hand side multiplied by L). This yields that $m_t^s \ge L \cdot i_t^s \Rightarrow \frac{m_t^s}{i_t^s} \ge L \quad \forall t \in \mathcal{T}$. Since $L = \frac{m_t^p}{i_t^p}$ from (17), this proves that constraint (13) holds. $\square$

4.3. K-bucket model $(\mathcal{P}^{kb})$

The L-bound model can be too conservative, because blocks sent from the stockpile with a grade greater than L are undervalued, and blocks with a grade lower than L cannot be sent to the stockpile to make up for this undervaluation. A better lower bound can be obtained assuming that we have several *buckets* of different grades. That is, we define K buckets, each of them with an associated minimum required grade $\hat{L}_k$, such that $\hat{L}_k \le \hat{L}_{k+1}$ for all $k = 1 \ldots K - 1$. Hence, a block that is sent to the stockpile can go to any bucket if it has the minimum required grade. We replace variables y_{bt}^s by variables $\bar{y}_{bkt}^s$, representing the fraction of each block sent to the kth-bucket of the stockpile. Then, constraint (8) is replaced by:

$$y_{bt}^p + \sum_{k \in \mathcal{K}} \bar{y}_{bkt}^s + y_{bt}^w = y_{bt}^m \quad \forall b \in \mathcal{B}, \forall t \in \mathcal{T} \tag{21}$$

Also, we track the material in each bucket by using variables $\bar{i}_{kt}^s$ and $\bar{i}_{kt}^p$ for each bucket k, replacing constraints (9) and (11) by

$$\bar{i}_{kt}^p \le \bar{i}_{k,t-1}^s \quad \forall t \in \mathcal{T} : t \ge 2, \forall k \in \mathcal{K} \tag{22}$$

$$\bar{i}_{kt}^s = \begin{cases} \sum_{b \in \mathcal{B}} W_b \bar{y}_{bkt}^s & t = 1 \\ \bar{i}_{k,t-1}^s - \bar{i}_{kt}^p + \sum_{b \in \mathcal{B}} W_b \bar{y}_{bkt}^s & \forall t \in \mathcal{T} : t \ge 2 \end{cases}, \forall k \in \mathcal{K} \tag{23}$$

and, as in the L-bound model $(\mathcal{P}^{lb})$, we replace constraint (13) by:

$$m_t^p = \sum_{k \in \mathcal{K}} \hat{L}_k \cdot \bar{i}_{kt}^p \quad \forall t \in \mathcal{T} \tag{24}$$

$$\bar{y}_{bkt}^s = 0 \quad \forall b \in \mathcal{B} \text{ such that } \frac{M_b}{W_b} < \hat{L}_k, \forall k \in \mathcal{K}, \forall t \in \mathcal{T} \tag{25}$$

where $\hat{L}_k$ is the pre-defined output grade of bucket k.

Note that with constraints (21)–(25) (see Table 1), the model is not a lower bound nor an upper bound of the nonlinear models $(\mathcal{P}^b)$ and $(\mathcal{P}^w)$. In fact, for $K = 1$, we recover the L-bound model with $L = \hat{L}_1$ because we have just one bucket in the stockpile; for $K = B$ and $\hat{L}_k = \frac{M_k}{W_k}$ for all $k = 1 \ldots K$, we recover the upper bound model because we have as many buckets as we have blocks. In order to obtain a lower bound on the objective function of the nonlinear model $(\mathcal{P}^b)$, we must add these constraints:

$$\bar{i}_{kt}^p = \bar{i}_{k't}^p \quad \forall k, k' \in \mathcal{K}, \forall t \in \mathcal{T} \tag{26}$$

$$\sum_{b \in \mathcal{B}} W_b \bar{y}_{bkt}^s = \sum_{b \in \mathcal{B}} W_b \bar{y}_{bk't}^s \quad \forall k, k' \in \mathcal{K}, \forall t \in \mathcal{T} \tag{27}$$

Lemma 2. *Let v^b be the optimal objective function value of the nonlinear model $(\mathcal{P}^b)$, and v^{kb} be that of model $(\mathcal{P}^{kb})$, then $v^{kb} \le v^b$.*

Proof. We show that from the optimal solution of $(\mathcal{P}^{kb})$ we can construct a feasible solution for $(\mathcal{P}^b)$. Define $i_t^p = \sum_{k=1}^K \bar{i}_{kt}^p$, $i_t^s = \sum_{k=1}^K \bar{i}_{kt}^s$ and $y_{bt}^s = \sum_{k=1}^K \bar{y}_{bkt}^s$. Constraint (2) is satisfied owing to the revision expressed in (21); constraints (9) and (11) are satisfied based on our variable definitions in which we cumulate over k. From constraint (24), we have $\forall t \in \mathcal{T}$:

$$m_t^p = \sum_{k=1}^K \hat{L}_k \cdot \bar{i}_{kt}^p$$

$$= \left(\sum_{k=1}^K \hat{L}_k \right) \cdot \bar{i}_{1t}^p \quad \text{(by (26))}$$

$$= \left(\frac{1}{K} \sum_{k=1}^K \hat{L}_k \right) \cdot \left(K \cdot \bar{i}_{1t}^p \right)$$

$$= \left(\frac{1}{K} \sum_{k=1}^K \hat{L}_k \right) \cdot i_t^p$$

where the last implication follows from the definition of i_t^p at the beginning of the proof and (26).

Hence, as in the proof of Lemma 1, this equality implies that

$$m_t^s = \sum_{b \in \mathcal{B}} \sum_{t' \leq t} M_b y_{bt'}^s - \sum_{t' < t} m_{t'}^p$$

$$= \sum_{b \in \mathcal{B}} \sum_{k=1}^{K} \sum_{t' \leq t} M_b \bar{y}_{bkt'}^s - \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \sum_{t' < t} i_{t'}^p$$

$$\geq \sum_{b \in \mathcal{B}} \sum_{k=1}^{K} \sum_{t' \leq t} \hat{L}_k W_b \bar{y}_{bkt'}^s - \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \sum_{t' < t} i_{t'}^p \quad \text{(by (25))}$$

$$= \sum_{k=1}^{K} \hat{L}_k \sum_{b \in \mathcal{B}} \sum_{t' \leq t} W_b \bar{y}_{bkt'}^s - \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \sum_{t' < t} i_{t'}^p$$

$$= \left(\sum_{k=1}^{K} \hat{L}_k \right) \cdot \left(\sum_{b \in \mathcal{B}} \sum_{t' \leq t} W_b \bar{y}_{b1t'}^s \right) - \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \sum_{t' < t} i_{t'}^p \quad \text{(by (27))}$$

$$= \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \cdot \left(K \cdot \sum_{b \in \mathcal{B}} \sum_{t' \leq t} W_b \bar{y}_{b1t'}^s \right) - \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \sum_{t' < t} i_{t'}^p$$

$$= \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \cdot \sum_{b \in \mathcal{B}} \sum_{t' \leq t} W_b y_{bt'}^s - \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \sum_{t' < t} i_{t'}^p$$

$$= \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \cdot \left(\sum_{b \in \mathcal{B}} \sum_{t' \leq t} W_b y_{bt'}^s - \sum_{t' < t} i_{t'}^p \right)$$

$$= \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) i_t^s \quad \text{(by (19))}$$

proving the result. □

4.4. L-average bound model ($\mathcal{P}^{la}$)

A novel way to approximate the nonlinear models similar to the L-bound is to fix the grade of the material leaving the stockpile to a fixed value L, but instead of requiring each block to have a grade greater than or equal to L, this model requires all the blocks that are going into the stockpile to have a grade greater than or equal to L "on average." Formally, we replace constraint (13) with the following constraints:

$$m_t^p = L \cdot i_t^p \quad \forall t \in \mathcal{T} \tag{28}$$

$$L \cdot \sum_{b \in \mathcal{B}} \sum_{t' < t} W_b y_{bt'}^s \leq \sum_{b \in \mathcal{B}} \sum_{t' < t} M_b y_{bt'}^s \quad \forall t \in \mathcal{T}. \tag{29}$$

Lemma 3. *Let v^b be the optimal objective function value of nonlinear model ($\mathcal{P}^b$), and v^{la} be that value of model ($\mathcal{P}^{la}$), then $v^{la} \leq v^b$.*

Proof. Let us reconsider the definitions of the variables i_t^s and m_t^s as given in constraints (19) and (20). We can multiply constraint (19) by L and use the following version of constraint (28): $\sum_{t' < t} m_{t'}^p = L \cdot \sum_{t' < t} i_{t'}^p$ to cancel the last term in (20) and the last term in (19) multiplied by L. Constraint (29) implies that the first term on the right hand side of (20) is greater than that same term in (19) multiplied by L. We can then compare left-hand sides of these same two equations to conclude that $m_t^s \geq L \cdot i_t^s$ for all $t \in \mathcal{T}$, proving that constraint (13) holds (because $L = \frac{m_t^p}{i_t^p}$ by constraint (28)). □

Moreover, we can prove that this model provides a better approximation to the nonlinear model ($\mathcal{P}^w$).

Lemma 4. *Let v^{lb}, v^{kb} and v^{la} be the optimal objective function values of models ($\mathcal{P}^{lb}$), ($\mathcal{P}^{kb}$) and ($\mathcal{P}^{la}$), respectively. For the best possible value of parameters L^{lb}, $\hat{L}_k^{kb}$ and L^{la}, respectively, the following inequalities hold:*

$$v^{lb} \leq v^{kb} \leq v^{la}$$

Proof. The first inequality holds because for any value of L^{lb} for ($\mathcal{P}^{lb}$), we can define two buckets, one with grade $\hat{L}_1 = L^{lb}$ and a second with grade $\hat{L}_2 \geq L^{lb}$. The optimal solution of model ($\mathcal{P}^{lb}$) can be reassigned to this 2-bucket model with a profit at least equal to v^{lb}.

For the second inequality, given an optimal solution of ($\mathcal{P}^{kb}$) with variables $\bar{y}_{bkt}^s$ representing the material in each bucket, if we define $y_{bt}^s = \sum_{k=1}^{K} \bar{y}_{bkt}^s$ and $\bar{L} = \frac{1}{K} \sum_{k=1}^{K} \hat{L}_k$, then y_{bt}^s satisfies constraint (29) for grade $L = \bar{L}$. In fact, by (25), we have that if $\bar{y}_{bkt}^s > 0$ then $M_b \geq \hat{L}_k \cdot W_b$. Hence,

$$\sum_{b \in \mathcal{B}} M_b y_{bt}^s = \sum_{b \in \mathcal{B}} \sum_{k=1}^{K} M_b \bar{y}_{bkt}^s \geq \sum_{b \in \mathcal{B}} \sum_{k=1}^{K} \hat{L}_k W_b \bar{y}_{bkt}^s$$

$$= \left(\sum_{k=1}^{K} \hat{L}_k \right) \cdot \left(\sum_{b \in \mathcal{B}} W_b \bar{y}_{b1t}^s \right) \quad \text{(by (27))}$$

$$= \left(\frac{1}{K} \sum_{k=1}^{K} \hat{L}_k \right) \cdot \left(K \cdot \sum_{b \in \mathcal{B}} W_b \bar{y}_{b1t}^s \right)$$

$$= \bar{L} \cdot \sum_{b \in \mathcal{B}} W_b y_{bt}^s$$

In other words, we can construct an equivalent feasible solution for model ($\mathcal{P}^{la}$) using $L = \bar{L}$ with the same objective, proving the result. □

4.5. Summary of all models

This section provides a summary of all models, listing the variables, the objective function and constraints associated with each model (see Table 1).

5. Graphical representations

We can assume that in a mine, some material is sent to the stockpile in the first time period and is processed at the mill in the second period. Since profit per ton is a linear function of the grade, we assume that the "grade" g of the material is defined by units of profit per ton. We can represent the total tonnage sent to the stockpile with grade greater than or equal to g using a function $G(g)$. An illustrative example of this function appears in Fig. 2. Note that if all of this material is sent to mill, the total profit recovered from the stockpile is equivalent to the area below $G(g)$.

In the case of the L-bound model, the total profit obtained from the stockpile is $L \cdot G(L)$, equivalent to the area of a rectangle below the curve (see Fig. 2a). Material with grade less than L cannot be sent to the stockpile in this model, and the material with grade at least L is extracted with a grade equal to L.

Fig. 2b represents, for the K-bucket model, the specific case of two buckets with grade L_1 and L_2, where $G(L_2)$ tons are extracted at profit L_2, and $G(L_1) - G(L_2)$ tons are extracted at grade L_1; we obtain a higher profit in this case than for the L-bound model, because the grade attributed to the material in the stockpile is more precisely matched with the true grade, and more material overall is allowed to be stockpiled. Note that constraint (26) requires that $G(L_1) - G(L_2) = G(L_2)$. Fig. 2c shows a selection for grade L_2 that improves the value of the material processed from the stockpile. Hence, the best selection of grades L_k for this example must satisfy the condition that $G(L_k) - G(L_{k+1}) = G(L_K)$ for $k = 1, \ldots, K - 1$. Fig. 2d shows an example with five buckets of equal tonnage. When K increases, a higher fraction of the total profit can be recovered from the stockpile.
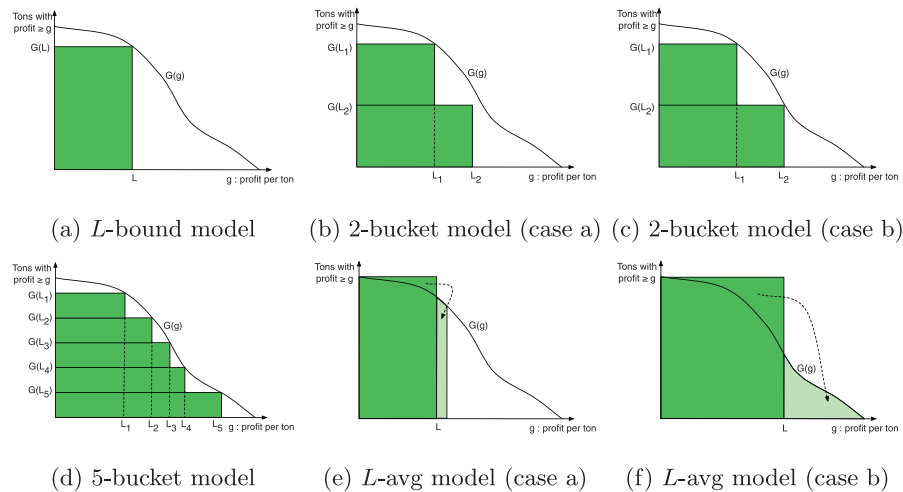
(a) *L*-bound model   (b) 2-bucket model (case a)   (c) 2-bucket model (case b)

(d) 5-bucket model   (e) *L*-avg model (case a)   (f) *L*-avg model (case b)

Fig. 2. Graphical examples of *L*-bound, *K*-bucket and *L*-average models. The green area shows the obtained profit by processing the material in the stockpile. The light green color shows the profit above $G(L)$ which compensates for the profit that was not obtained below $G(L)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Table 2
Problem instance characteristics.

Instance	# blocks	# blocks in ultimate pit	Discount factor (%)	Mining cap (MT/year)	Mill cap (MT/year)
newman1	1,060	1,059	8	2	1.1
marvin	53,271	8,516	10	60	20.0
zuck-small	9,400	9,399	10	60	20.0
sm2	99,014	18,388	10	12	1.4, 1.8, 2.1, 2.2,...,2.2
zuck-medium	29,277	27,387	10	18	8.0
zuck-large	96,821	96,821	10	3	1.2

Fig. 2e shows an example of the *L*-average model in which we can extract more than $G(L)$ tons with grade *L*. This is possible because the green area above the curve is compensated for by the profit of material with a grade greater than *L* (light green region). The figure shows that we can increase the value of *L*, allowing us to obtain a higher profit by extracting the same material. In fact, there exists a value L^* such that we gain the maximal profit, equivalent to the area below $G(L)$ (see Fig. 2f).

6. Computational experiments

In this section, we examine the solution quality associated with different linear-integer and nonlinear-integer models. In Section 6.1, we compare the proposed models to the nonlinear model. This requires a reduction in problem size, accomplished by fixing the block extraction time in all models. In Section 6.2, we compare two linear-integer models using a customized solver called OMP (Rivera, Brickey, Espinoza, Goycoolea, & Moreno, 2016) without fixing the block extraction time. Unless otherwise stated, we perform all computation on a Dell R620 with eight Xeon E5-2670 2.0 gigahertz cores and 128 gigabytes RAM.

6.1. Comparing different linear models to the nonlinear model

Our first computational experiment compares the quality of our linear-integer models against that of the nonlinear models presented by Bley et al. (2012a). We coded the models in AMPL (2014) and solved models $(\mathcal{P}^{ns})$, $(\mathcal{P}^{ub})$, $(\mathcal{P}^{la})$, $(\mathcal{P}^{kb})$, and $(\mathcal{P}^{lb})$ using CPLEX (2009). Nonlinear models $(\mathcal{P}^b)$ and $(\mathcal{P}^w)$ were solved using SCIP 3.1.0 (Vigerske & Gleixner, 2016) with CPLEX 12.6 as the linear solver.

We use six instances: *newman1, marvin, sm2, zuck-small, zuck-medium*, and *zuck-large*, available on the Minelib website (Espinoza,

Goycoolea, Moreno, & Newman, 2013). Table 2 presents the unique characteristics of these instances, which include two capacity constraints for each time period: one for the total mined material, and a second for the total processed material; the latter restriction makes inventory relevant. We add one stockpile using a rehandling cost equal to 10% of the original mining cost. Standard solvers are not able to produce optimal solutions for such large instances, even without stockpiles. Hence, in order to compare solver performance more precisely, i.e., through an optimal objective function value, we simplify our instances by fixing the extraction time period (i.e., the *x*-variables) in all models to that of the best-known solution for each instance presented on the Minelib website. Note that resulting problems have only continuous variables. We later relax this assumption.

Bley, Gleixner, Koch, and Vigerske (2012b) explain that the nonlinear model $(\mathcal{P}^w)$ performs considerably better than $(\mathcal{P}^b)$, but requires a substantial amount of memory, e.g., we are only able to obtain solutions to $(\mathcal{P}^w)$ for instances *newman1, zuck-small, sm2* and *marvin* due to memory requirements; the latter instance requires more than 100 gigabytes of RAM to obtain a solution within a 0.13% optimality gap after two weeks of run time.

On the contrary, we found the optimal solution using each of the linear models $(\mathcal{P}^{lb})$, $(\mathcal{P}^{kb})$ and $(\mathcal{P}^{la})$ in a few seconds of run time for all instances. For models $(\mathcal{P}^{la})$ and $(\mathcal{P}^{lb})$, we tested several values of *L* with a view to improving the objective function for which our numerical experiments indicate unimodality in *L*, enabling us to perform a simple line search for its optimal value corresponding to each model and instance. In the case of $(\mathcal{P}^{kb})$, for ease of comparison, we consider four buckets for all instances and explore several bucket *k* grade values L_k under the assumption that by increasing the number of buckets in the stockpile, the objective function value of $(\mathcal{P}^{kb})$ approaches the upper bound of the problem given by $(\mathcal{P}^{ub})$.

Table 3
NPV normalized to $(\mathcal{P}^{ns})$ model.

Instance	Original mill capacity					50% of original mill capacity				
	$(\mathcal{P}^{ub})$	$(\mathcal{P}^{w})$	$(\mathcal{P}^{la})$	$(\mathcal{P}^{kb})$	$(\mathcal{P}^{lb})$	$(\mathcal{P}^{ub})$	$(\mathcal{P}^{w})$	$(\mathcal{P}^{la})$	$(\mathcal{P}^{kb})$	$(\mathcal{P}^{lb})$
newman1	1	1	1	1	1	1.1957	1.1858	1.1857	1.1145	1.0959
marvin	1.0212	1.0170	1.0169	1.0114	1.0081	1.2823	1.2246	1.2226	1.1647	1.1420
zuck-small	1.0199	1.0157	1.0157	1.0104	1.0073	1.2735	1.2165	1.2132	1.1548	1.1342
sm2	1.0025	1.0019	1.0016	1.0011	1.0005	1.2589	–	1.1406	1.0749	1.0693
zuck-medium	1.0159	–	1.0126	1.0086	1.0062	1.3237	–	1.2429	1.1608	1.1451
zuck-large	1.0061	–	1.0047	1.0031	1.0029	1.1883	–	1.1262	1.1013	1.0935

Table 3 displays the resulting NPV, normalized to that of the $(\mathcal{P}^{ns})$ model. The objective function values corresponding to all models for the *newman1* instance are the same because we have extra mill capacity and therefore there is no incentive to stockpile. The differences in objective function values between the most extreme models, models $(\mathcal{P}^{ns})$ and $(\mathcal{P}^{ub})$, for the *marvin* and *zuck-small* instances (i.e., those that benefit most from stockpiling) are 2.07% and 1.95%, respectively. For the *newman1*, *marvin*, *sm2* and *zuck-small* instances, that difference between $(\mathcal{P}^{w})$ and $(\mathcal{P}^{la})$, the theoretical nonlinear "mixing model" and our closest approximation to it, is less than 0.17%, and the difference between $(\mathcal{P}^{w})$ and $(\mathcal{P}^{kb})$ is less than 0.7%. In other words, the $(\mathcal{P}^{la})$ model provides a very close approximation to the objective function given by $(\mathcal{P}^{w})$.

A comparison of the objective function values from the models whose instances can all be solved in our numerical experiments indicate a difference between $(\mathcal{P}^{ub})$ and $(\mathcal{P}^{kb})$ of less than 0.97%, and a difference between $(\mathcal{P}^{ub})$ and $(\mathcal{P}^{la})$ of less than 0.42%. In other words, these linear models approximate the nonlinear model very well, provide solutions with corresponding objective function values that are close to the theoretical optimum, and are much more tractable.

Changing the maximum mill production capacity illustrates that there is some trade-off between stockpiling and this parameter. In order to better demonstrate the difference between our proposed models, we decrease the milling capacity relative to the mining capacity, which increases the value of the stockpile, especially with a fixed extraction sequence, and a corresponding relative increase in the amount of material left on the stockpile relative to what is extracted. Table 3 illustrates the trade-off between stockpiling and mill capacity for all instances. Decreasing mill capacity results in as much as 20% value added at half of the original for the *newman1* instance. Relative to each other, the models perform similarly; the difference in objective function value between $(\mathcal{P}^{w})$ and $(\mathcal{P}^{la})$ is less than 0.03% for the three instances that we were able to solve.

6.2. Comparing linear-integer models considering the extraction sequence

Our set of computational experiments in Section 6.1 shows that $(\mathcal{P}^{la})$ provides a very close objective function value to that of $(\mathcal{P}^{w})$ for a model with a predefined extraction sequence. In this section, we first compare the quality of the *LP relaxation* provided by $(\mathcal{P}^{ns})$, $(\mathcal{P}^{ub})$, and $(\mathcal{P}^{la})$ without fixing the extraction time. Note that state-of-the art solvers based on the Bienstock–Zuckerberg algorithm only solve the corresponding linear program of our linear-integer production scheduling problems in an exact way, and then apply other techniques that use this relaxed solution to generate a near-optimal integer solution, e.g., the academic solver OMP. The nonlinear model cannot be solved with the current state-of-the-art algorithms, e.g., SCIP and BARON, even with the size of machine we use. Therefore, we focus on the difference between the LP-relaxation of our *L*-average and the upper bound models, because they can be computed in an exact way.

Table 4
NPV normalized to $(\mathcal{P}^{ns})$ for the original mill capacity case without fixing the extraction time, and solved with OMP.

Instance	$(\mathcal{P}^{ub})$	$(\mathcal{P}^{la})$
newman1	1	1
marvin	1.0500	1.0365
zuck-small	1.0516	1.0377
sm2	1.0087	1.0047
zuck-medium	1.0489	1.0373
zuck-large	1.0108	1.0091

Table 4 displays the resulting NPV, normalized to that of the no-stockpile model. The time required by OMP to solve these problems varies from a few seconds (*newman1*) to 19 minutes (*zuck_large*). Because of the limitations of the OMP solver, and for the sake of consistency between models, for these instances, we do not consider rehandling costs. The upper bound model and the solution provided by the *L*-average model differ in objective function value by less than 0.9%, on average, with a difference for the *zuck-small* instance of 1.4%. These ranges show that the *L*-average model provides upper bounds that are a good approximation to those provided by the nonlinear model for the more general case in which we include extraction decisions.

Finally, we show that standard rounding techniques, e.g., TopoSort (Chicoisne et al., 2012), still provide good integrality gaps for the mixed-integer version of the $(\mathcal{P}^{la})$ model. Table 5 shows the objective value of the LP relaxation compared with that from the integer solution obtained by running TopoSort considering a stockpile, i.e., using model $(\mathcal{P}^{la})$ and without considering a stockpile, i.e., using model $(\mathcal{P}^{ns})$. Standard rounding heuristics yield near-optimal solutions for both models, demonstrating that the additional variables and constraints required to model a stockpile do not loosen the LP relaxation, a crucial property for solving large-scale instances of the problem. In fact, this is not true for the $(\mathcal{P}^{ub})$ model.

All the models we present can be easily adapted to the case of scheduling clusters of blocks (e.g., predefined bench-phases, bins or panels) by redefining extraction binary variables x_{bt}. Specifically, we can replace these variables by x_{ct} for clusters $c \in C$ of blocks, and modify (4) and (5) such that these constraints apply to each block in its corresponding cluster. In this way, the number of binary variables can be reduced considerably, making it possible to obtain optimal integer solutions using a Bienstock–Zuckerberg algorithm embedded in a branch-and-bound scheme, e.g., within the spatial branching proposed by Bley et al. (2012a), enabling us to solve these instances exactly; however, doing so is outside of the scope of this paper.

7. Conclusion

Considering stockpiling as part of open pit mine planning presents numerous challenges: (i) the most precise model in the literature at the time of this writing is nonlinear and integer,

Table 5
Objective function values of the LP relaxation (LP), the integer program obtained via TopoSort (IP), and the corresponding integrality gaps (Int. Gap) obtained with OMP for our best stockpiling model, $(\mathcal{P}^{la})$, and the corresponding model with no stockpiling, $(\mathcal{P}^{ns})$.

Instance	LP $(\mathcal{P}^{la})$	IP $(\mathcal{P}^{la})$	Int. Gap (%)	LP $(\mathcal{P}^{ns})$	IP $(\mathcal{P}^{ns})$	Int. Gap (%)
newman1	24.486	24.486	0.000	24.486	24.486	0.000
marvin	944.767	935.024	1.031	911.480	889.729	2.386
zuck-small	939.728	927.599	1.291	905.544	879.886	2.833
sm2	1660.120	1658.475	0.099	1652.393	1650.818	0.095
zuck-medium	776.037	718.464	7.419	748.150	710.379	5.049
zuck-large	58.468	57.891	0.986	57.938	57.391	0.945

yielding a non-convexity and therefore no guarantee of a global optimum; (ii) nonlinear-integer models are often intractable, especially for realistically sized instances; (iii) even if we obtain a solution for these models, the way in which some assumptions are handled, in particular, that of homogeneous mixing of the material in a single stockpile in each time period, is unrealistic. This paper proposes several variants of linear-integer models that expedite solutions. Computational experiments show that the linear-integer model, $(\mathcal{P}^{la})$, the best for the realistic instances we test, is tractable and possesses an objective function value very close to that of $(\mathcal{P}^{w})$ and $(\mathcal{P}^{b})$, the nonlinear models.

Blending ore with contaminants can be modeled with $(\mathcal{P}^{la})$; in this case, the economical impact of stockpiles could be considerably higher than in our cases. The proposed L-average model can easily be extended to consider more than one grade, and to account for degradation in the stockpile. In practice, since model instances solve sufficiently quickly, it is possible to iteratively determine an optimal value of L using a binary search. However, an interesting avenue for future research would integrate the branching scheme proposed by Bley et al. (2012a) to this end, which would prove to be of particular importance for the trivial extension of the L-average model to the case in which a different value of L exists for each time period.

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