

# Dynamic of cyclic automata over $\mathbb{Z}^2$

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## Abstract

Let  $K$  be the two-dimensional grid. Let  $q$  be an integer greater than one and let  $Q = \{0, \dots, q-1\}$ . Let  $s : Q \rightarrow Q$  be defined by  $s(\alpha) = (\alpha + 1) \bmod q, \forall \alpha \in Q$ .

In this work we study the following dynamic  $F$  on  $Q^{\mathbb{Z}^2}$ . For  $x \in Q^{\mathbb{Z}^2}$  we define  $F_v(x) = s(x_v)$  if the state  $s(x_v)$  appears in one of the four neighbors of  $v$  in  $K$  and  $F_v(x) = x_v$  otherwise.

For  $x \in Q^{\mathbb{Z}^2}$  such that  $\{v \in \mathbb{Z}^2 : x_v \neq 0\}$  is finite we prove that there exists a finite family of cycles such that the period of every vertex of  $K$  divides the lcm of their lengths. This is a generalization of the same result known for finite graphs. Moreover, we show that this upper bound is sharp. We prove that for every  $n \geq 1$  and every collection  $k_1, \dots, k_n$  of non-negative integers there exists  $y^n \in Q^{\mathbb{Z}^2}$  such that  $|\{v \in \mathbb{Z}^2 : y_v^n \neq 0\}| = O(k_1^2 + \dots + k_n^2)$  and the period of the vertex  $(0, 0)$  is  $p \cdot \text{lcm}\{k_1, \dots, k_n\}$ , for some even integer  $p$ .

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## 1 Introduction

Let  $G = (V, E)$  be a non directed graph and let  $Q = \{0, 1, \dots, q-1\}$  be where  $q$  is a non negative integer with  $q \geq 2$ . A  $Q$ -state on  $G$  is a vector  $(x_v : v \in V)$  such that  $x_v \in Q$  for all  $v \in V$ .

In this work we study the dynamical system  $(Q^V, F)$  where  $F$  is given as follows. Let  $x$  be a  $Q$ -state of  $G$ . For every  $v \in V$ , the state  $F_v(x)$  is given according to the next-cyclic neighbor (NCN) rule: if the state  $s(x_v) := x_v +$

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1 mod  $q$  is the state of a neighbor of  $v$  then the new state of  $x_v$  is  $s(x_v)$ . Otherwise, the state of vertex  $v$  does not change.

The convergence to a global state where all the vertices have the same state was studied in [Fis90]. There, the initial state is defined by assigning to each vertex a state chosen uniformly random from the set  $Q$ . This model was also previously studied in [BG89] as a Markov process in continuous time with stochastic evolution, that is, the time difference between two consecutive steps is a random variable with exponential distribution. In both works the model studied was over unidimensional and bi-dimensional grids.

For  $q = 2$  this process becomes the voting model, studied in [CS73] and [HL75]. For further references see [FGG91, Fis92, Fis91].

In [MG97] this dynamic was studied in finite graphs, when all the vertices update their states synchronously in discrete times. Obviously, for finite graphs the sequence of global states generated by the dynamic eventually becomes periodic. The main result in [MG97] is that the length of this period is upper bounded by the l.c.m. of the lengths of some *special* cycles of the graph. Moreover, for every non negative integer  $n$  it is constructed a graph  $G_n$  with  $p(n)$  vertices, for some polynomial  $p$ , with  $n$  special cycles of lengths  $4, 8, \dots, 4n$  and such that the period of the dynamic is at least  $4 \cdot \text{lcm}\{1, 2, \dots, n\}$ . Therefore, for general graphs the upper bound is the best possible.

In this work we study the NCN rule in the 2-dimensional lattice  $K$ . This is an infinite graph where each vertex has four neighbors: left, right, up and down. Given an  $Q$ -state on  $K$  the state  $p$  of a vertex  $u$  changes if and only if some of its neighbor  $v$  has the state  $s(p)$ . Therefore, the set of all the edges  $uv$  connecting vertices in states  $p$  and  $p'$  with  $p = s(p')$  or  $p' = s(p)$  completely determines the next states of all the vertices in  $K$ . For technical reasons the edges connecting vertices with the same state are also relevant. That leads us to the following definition.

The skeleton of  $K$  with respect to an  $Q$ -state  $x$  is the subgraph of  $K$  defined as follows. Its set of vertices is the set of vertices of  $K$  and its edges are those edges of  $K$  connecting vertices in the states  $p$  and  $p'$  with  $p = p'$  or  $p = s(p')$  or  $p' = s(p)$ .

In section 3, we study the dynamic of NCN rule when the skeleton is the whole graph  $K$ . For  $q \geq 5$  we prove that the period at any vertex is 1, that is, after a finite time the state of the vertex remains constant. For  $q = 4$  we show that either the period of all vertices is 1 or the period of all vertices is four. We also prove that when  $q = 3$  the dynamic is richer: for every even integer  $t \geq 14$  there exists a  $Q$ -state on  $K$  with at most  $t$  states different from zero such that the period of every vertex is exactly  $t$ .

In section 4, we present the main contribution of the paper. We prove that for every  $q \geq 4$  and for every non negative integer  $n$  there exists a  $Q$ -state on  $K$  such that there is a vertex that has period at least  $4 \cdot \text{lcm}\{1, 2, \dots, n\}$ . Moreover, the  $Q$ -state has at most  $p(n)$  states different from zero, where  $p$  is a polynomial function of  $n$ . To prove this result we embed a construction given in [MG97] into the graph  $K$ . We also prove that the upper bound for the period in finite graphs remains valid in the infinite graph  $K$  when the  $Q$ -state has a finite number of non-quiescent states.

In Section 2 we present the known dynamical results about NCN dynamic in finite graphs which are necessary to present our results.

## 2 Definition of the Problem

We give the main definitions used in NCN. The notation used throughout the paper is consistent with those from the book [Die00].

Let  $G$  be a graph and let  $x$  be a  $Q$ -state on  $G$ . The *skeleton*  $Es(G, x) = (V, E(x))$  of  $x$  on  $G$  is the spanning subgraph of  $G$  whose set of edges is defined as follows. An edge  $e = uv$  belongs to  $E(x)$  if and only if the states  $x_u$  and  $x_v$  are *adjacent* in  $Q$  that is,  $x_u = x_v$  or  $x_u = s(x_v)$  or  $x_v = s(x_u)$ . The *skeleton* of a  $Q$ -state  $x$  on  $K$  is denote  $Es(x)$ . For a given  $Q$ -state  $x$  on  $K$  we define the *skeleton* of  $x$  (on  $K$ ) as the spanning subgraph  $Es(x) = (\mathbb{Z}^2, E(x))$  of  $K$  whose set of edges is defined as follows. An edge  $e = uv$  belongs to  $E(x)$  if and only if the states  $x_u$  and  $x_v$  are *adjacent* in  $Q$  that is,  $x_u = x_v$  or  $x_u = s(x_v)$  or  $x_v = s(x_u)$ .

We say that a  $Q$ -state  $x$  on  $K$  has finite support if the cardinality of the set

$$\text{supp}(x) := \{v \in \mathbb{Z}^2 : x_v \neq 0\}$$

is finite. In the sequel a  $Q$ -state will refer to a  $Q$ -state of finite support.

For a  $Q$ -state of finite support the set of edges not in the skeleton of  $x$  is finite. Moreover, if  $F(x)$  is the  $Q$ -state obtained from  $x$  by applying the NCN rule then  $Es(x) \subseteq Es(F(x))$  (see Lemma 1 in [MG97]). Hence, there is a time  $t^*$  such that  $Es(F^{t^*+i}(x)) = Es(F^{t^*}(x)) \forall i$  that is, after time  $t^*$  the skeleton becomes stable.

We need some definitions before to describe the generalization of the known dynamical properties of NCN from finite to infinite graphs. Let  $G = (V, E)$  be a graph. A *walk*  $P$  in  $G$  is a sequence  $v_0, \dots, v_k$  of vertices of  $G$  such that  $v_i$  is adjacent with  $v_{i+1}$  for  $i = 0, \dots, k - 1$ . It is a *closed walk* if  $v_0 = v_k$ . The *length* of  $P$  is  $k - 1$  and it is denoted by  $L(P)$ . If all the vertices  $v_i$  are different

then  $P$  is called a *path*. If we add the edge  $v_0v_k$  to a path  $v_0, \dots, v_k$  we obtain a *cycle*.

For two states  $\alpha$  and  $\beta$  adjacent in  $Q$  we define the jump  $\mu(\alpha, \beta)$  by  $\mu(\alpha, \beta) = 1$  if  $s(\alpha) = \beta$ ,  $\mu(\alpha, \beta) = -1$  if  $s(\beta) = \alpha$  and  $\mu(\alpha, \beta) = 0$  otherwise.

Let  $x$  be a  $Q$ -state on  $G$ . Let  $P = (v_0, \dots, v_k)$  be a walk in the skeleton  $Es(G, x)$ . The jump  $J(P, x)$  of  $P$  on  $x$  is the sum of the jumps  $\mu(x_{v_i}, x_{v_{i+1}})$ , for  $i = 0, \dots, k-1$ . In this definition the order of the vertices in  $P$  is relevant. Indeed, if  $-P = (v_k, \dots, v_0)$  then  $J(-P, x) = -J(P, x)$ . Notice that if  $P$  is a closed walk then  $q$  divides  $J(P, x)$ .

For an integer  $L \geq 1$  and a vertex  $v \in V$  let  $a(v, L, x)$  the maximum jump among all the jumps (on  $x$ ) of walks starting in  $v$  and having lengths less or equal than  $L$ . Let  $h(v, L, x)$  be the minimum length among all the lengths of walks starting in  $v$  and having maximum jump  $a(v, L, x)$ .

It is easy to see that the next proposition proved in [MG97] for finite graph remains valid in  $K$  (actually in every infinite graph finitely locally connected)

**Proposition 1** *Let  $x$  be a  $Q$ -state such that  $Es(F(x)) = Es(x)$ . Then for all  $L \geq 1$  we have.*

- (1) *If  $a(v, L, x) \leq 0$  then  $F_v^l(x) = x_v, \forall l \in \{1, \dots, L\}$ .*
- (2) *If  $a(v, L, x) > 0$  then  $F_v^l(x) = s^{a(v, L, x)}(x_v), \forall l \in \{h(v, L, x), \dots, L\}$ .*

Let  $x$  be a  $Q$ -state with  $Es(F^i(x)) = Es(x)$ , for all  $i$ . Let  $G = (V, E)$  be a finite subgraph of  $K$  containing all the elements in  $\text{supp}(x)$  and such that for all  $u \notin V$  all the edges incident with  $u$  belong to  $E(x)$ . Then the values  $a(v, L, x)$  and  $h(v, L, x)$  on  $K$  can be computed by considering only walks inside  $G$ . From Proposition 1 we know that the evolution of any vertex is completely determined by the values  $a(v, L, x)$  and  $h(v, L, x)$ . Then, in order to know the evolution of a vertex  $u \in V$  under the function  $F$  defined on  $K$  it is enough to consider its evolution under  $F$  defined on  $G$ .

Let  $P$  be a walk in  $Es(x)$ . The efficiency  $e(P, x)$  of  $P$  on  $x$  is the quotient between its jump and its length. Since  $|J(P, x)| \leq L(P)$  we have  $|e(P, x)| \leq 1$ .

Intuitively, the efficiency of a walk measures the potential that the walk has to impose the states of its vertices to its initial vertex. By instance, if two walks  $P_1$  and  $P_2$  starting at the same vertex  $v$  have both jump  $a(v, L, x)$  but  $L(P_1) < L(P_2)$  then  $P_1$  (the most efficient) would impose its states to its initial vertex.

For a finite graph  $G = (V, E)$  and a  $Q$ -state  $x$  on  $G$  such that  $Es(F^i(x)) = Es(x), \forall i$  it is clear that for every  $u \in V$  the sequence of states  $(F_u^t(x))_{t \geq 0}$  is

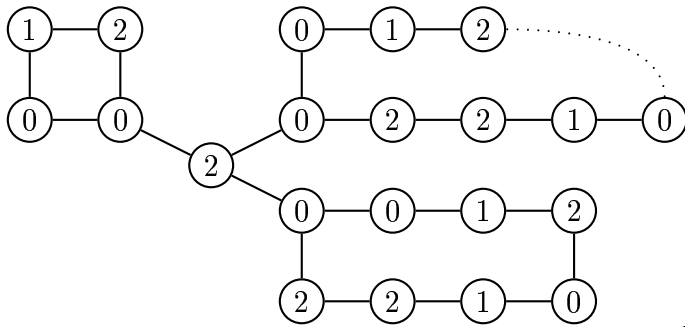


Fig. 1. NCN that reaches the bound  $T(x) = e^{\Omega(\sqrt{|V|})}$  (see [MG97].)

eventually periodic. Moreover, it is known that its period divides the lcm of the lengths of cycles with maximum efficiency in  $G$  (see [MG97]). Let  $T(x)$  be the period of the NCN rule on  $K$  with initial condition  $x$  defined as the supremum over all the periods of vertices of  $K$ . From previous analysis we know that  $T(x)$  is finite. Moreover, it divides the lcm of the lengths of cycles with maximum efficiency in  $Es(F^{t_0}(x))$ , where  $t_0 := 4|\text{supp}(x)|$  since  $Es(F^{t_0+1}(x)) = Es(F^{t_0}(x))$ . Let  $e$  be the maximum of the efficiencies of cycles in  $Es(F^{t_0}(x))$ . We have proved the following result.

**Theorem 2** *For every  $Q$ -state  $x$  the period of the NCN rule on  $K$  is finite. Moreover, it divides the lcm of the lengths of cycles in  $Es(F^{t_0}(x))$  with efficiency  $e$ .*

The upper bound for the period of NCN rule in finite graphs has proved to be asymptotically optimal: there is an infinite family of finite graphs  $(G_n)_{n \geq 1}$  such that for each  $n$  there is a  $Q$ -state  $y^n$  on  $G_n$  such that the period of the NCN rule is the lcm of the lengths of cycles with maximum efficiency on  $y^n$ . The graph  $G_n$  consists of  $n$  cycles  $C_k$ ,  $k = 1 \dots n$ , joined by a central node  $v_0$ . The length of the cycle  $C_k$  is  $4k$ . The  $Q$ -state  $y^n$  on  $C_k$  is  $0012(0112)^{k-1}$ , starting in the vertex of  $C_k$  which is the neighbor of the central vertex  $v_0$ . Therefore, all cycles have efficiency  $\frac{3}{4}$  and it is the maximum efficiency on  $G$ . Finally, the state of  $v_0$  is set to 2.

Since every cycle evolves independently, the sequence of states in the central vertex will be  $0012(0112)^{u-1}$  where  $u = \text{lcm}\{4k : k = 1 \dots n\}$  (see Figure 1).

We shall provide a construction of  $Q$ -states  $y^n$  on  $K$ , for  $n \geq 1$  such that the period of the NCN rule on  $K$  is the lcm of the lengths of cycles with maximum efficiency on  $y^n$ . The  $Q$ -state  $y^n$  will be an embedding on  $K$  of the graph  $G_n$  previously described. It is not hard to see that if the period of the NCN is greater than 1 all the vertices in  $K$  will have period greater than 1. This implies that the stable skeleton is connected even if it is a proper subgraph of  $K$ . Therefore, the main problem we have to solve in order to construct the  $Q$ -state  $y^n$  on  $K$  is the interaction between the cycles due to the connectness of the stable skeleton. To get inside of the problem we devote the next section

to the study of fully connected skeleton and we postpone the construction of  $y^n$  to Section 4.

### 3 Complete Skeletons

In this section, we study the dynamic of NCN rule when the skeleton is the whole graph  $K$ . For  $q \geq 5$  we prove that the period at any vertex is 1, that is, after a finite time the state of the vertex remains constant. For  $q = 4$  either the period of all vertices is 1 or the period of all vertices is four. We also prove that when  $q = 3$  the dynamic is richer: for every even integer  $t \geq 14$  there exists an  $Q$ -state on  $K$  with at most  $t$  states different from zero such that the period of every vertex is exactly  $t$ .

The basic units in the graph  $K$  are cycles of length four called *tiles*. It is clear that a tile in a skeleton can only have efficiency 0,  $\frac{3}{4}$  or 1. Using a decomposition property we will see that a similar property holds for every cycle.

For two cycles  $\gamma_1$  and  $\gamma_2$  whose intersection is a path  $P = i_k, \dots, i_l$  we define their *union*  $\gamma_1 \sqcup \gamma_2$  as the cycle  $i_k \gamma_1 i_l \gamma_2 i_k$  containing no inner vertex of  $P$  and no inner edge of  $P$ .

We have the following lemma.

**Lemma 3** *Let  $x$  be an  $Q$ -state over  $G$ . Let  $\gamma_1 = i_0 i_1 \dots i_n$  and  $\gamma_2 = j_0 j_1 \dots j_{n'}$  be two cycles such that  $\gamma_1 \cap \gamma_2$  is a path  $P$  over  $K$ . Given a direction for  $\gamma := \gamma_1 \sqcup \gamma_2$  and the induced directions over  $\gamma_1$  and  $\gamma_2$  we have*

$$J(\gamma, x) = J(\gamma_1, x) + J(\gamma_2, x)$$

**PROOF.** The jump of  $\gamma$  is the addition of the jump of  $\gamma_1 \setminus \overset{\circ}{P}$  and the jump of  $\gamma_2 \setminus \overset{\circ}{P}$  where  $\overset{\circ}{P}$  denotes the set of inner vertices of  $P$ . Since the path  $P$  is traversed in opposite direction by  $\gamma_1$  and  $\gamma_2$ , the jump of  $P$  in  $\gamma_1$  (denoted  $J_1$ ) is minus the jump of  $P$  in  $\gamma_2$  ( $J_2$ ). Therefore, the jump of  $\gamma_1 = \gamma_1 \setminus \overset{\circ}{P} + J_1$  and  $\gamma_2 = \gamma_2 \setminus \overset{\circ}{P} + J_2$ . Finally, the jump of  $\gamma$  is equal to the jump of  $\gamma_1$  plus the jump  $\gamma_2$ .  $\square$

It is easy to see that the following property holds.

**Corollary 4** *If  $\gamma_1$  and  $\gamma_2$  are two cycles of efficiencies 0 then the cycle (if ever)  $\gamma_1 \sqcup \gamma_2$  has efficiency 0.*

Previous properties are true for every cycle, in particular for tiles.

**Lemma 5** *Let  $q \geq 3$  be an integer and let  $x$  be a  $Q$ -state over  $K$ . If every tile has jump zero then the efficiency of the system  $e(x)$  is zero.*

**PROOF.** We prove that the jump of every cycle in  $K$  is zero. Let  $\gamma$  be a cycle in  $K$ . We prove the property by induction on the number  $n$  of inner points  $(i + \frac{1}{2}, j + \frac{1}{2})$ ,  $i, j \in \mathbb{Z}$  in the bounded region  $C$  of  $\mathbb{R}^2 \setminus \gamma$ . The case  $n = 1$  is when  $\gamma$  is a tile, by hypothesis its jump is zero.

Let us assume that the properties hold for every cycle with less than  $n$  inner points. Let  $v$  be any vertex in  $\gamma$  such that at least one incident edge of  $v$  in  $K$  has a part in the region  $C$ . We cut the region  $C$  along a straight line starting in  $v$  and ending in a vertex of  $\gamma$  and having no other vertex in common with  $\gamma$ . This straight line defines a path in  $K$ . With this path and the cycle  $\gamma$  we build two cycles  $\gamma'$  and  $\gamma''$  with less inner points than  $\gamma$  such that  $\gamma' \cup \gamma'' = \gamma$ . By induction hypothesis we know that the jump of  $\gamma'$  and  $\gamma''$  is zero. Therefore, from the Lemma 3 we conclude that the jump of  $\gamma$  is zero.  $\square$

**Lemma 6** *Let  $q$  be an integer. Let  $x$  be a  $Q$ -state over  $K$  such that  $Es(x) = K$ .*

- (1) *If  $q \geq 5$  then  $e(x) = 0$ .*
- (2) *If  $q = 4$  then  $e(x) \in \{0, 1\}$ .*
- (3) *If  $q = 3$  then  $e(x) = 0$  or  $e(x) \geq \frac{3}{4}$ .*

**PROOF.** Let  $\alpha$  be the maximum jump over all the tiles. If  $\alpha = 0$  then from previous lemma we conclude. Since for  $q \geq 5$  all tiles have jump zero, the only cases where  $\alpha \neq 0$  are  $q = 3, 4$ .

- (1) Case  $q = 4$ . If  $\alpha > 0$  then at least one tile  $T$  has jump 4. Therefore the tile  $T$  has efficiency 1 which is obviously the maximum efficiency of the system.
- (2) Case  $q = 3$ . If  $\alpha > 0$  then at least one tile  $T$  has jump 3. Hence the tile  $T$  has efficiency  $\frac{3}{4}$ . Therefore, the maximum efficiency will be greater or equal to  $\frac{3}{4}$ .

$\square$

In order to get richer dynamics we can either consider complete skeleton and  $q = 3$  or to allow skeletons to be proper subgraph of  $K$ . Since we want to know the result for arbitrary  $q$  we have performed our constructions in the

second framework. Nevertheless, we have proved that at least in term of the efficiency the case  $q = 3$  is rich enough as stated in the following theorem.

**Theorem 7** *Let  $q$  be an integer.*

- (1) *If  $q \geq 5$  then the period of any vertex is 1 for all  $Q$ -state  $x$  with  $Es(x) = K$ .*
- (2) *If  $q = 4$  then the period of any vertex is 1 or 4 for all  $Q$ -state  $x$  with  $Es(x) = K$ .*
- (3) *If  $q = 3$ , for any even integer  $t \geq 14$  there exists an  $Q$ -state  $x$  with  $Es(x) = K$  such that the period of any vertex is  $t$ . Moreover, this  $Q$ -state has at most  $t$  vertices with state different from zero.*

**PROOF.** The two first cases are directly from the previous lemma. For  $q = 3$  we construct a rectangular cycle  $R$  with sides of lengths  $\lfloor \frac{t}{4} \rfloor$  and  $\lceil \frac{t}{4} \rceil$  with the following sequence over his vertices:  $i$  times 012 and  $j$  times 0112 for any  $i, j > 0$  such that  $3i + 4j = t$ . We will prove that this is the unique cycle of maximum efficiency in this  $Q$ -state, and how the skeleton is complete, its period  $t$  will be the period of any vertex.

Indeed, a cycle sharing no edge with  $R$  has jump 0. Each cycle  $S$  sharing two or more paths with  $R$  can be replaced by a shorter cycle  $S'$  which shares only one path with  $R$  and with a bigger jump. So, the cycle  $S$  can be decomposed in two paths  $S_R$  and  $S_I$  such that  $S_R$  is a path inside  $R$  and  $S_I$  is a path outside  $R$ . Clearly, the subpath of  $S$  outside  $R$  is a shortest path between its ends. It can be seen that if the ends of the path  $S_I$  are not in opposite sides of  $R$  then the efficiency of  $S$  is either smaller than  $\frac{1}{2}$  or smaller than the efficiency of  $R$  (smaller jump and more or equal length). If  $S_I$  is neither vertical nor horizontal but with both ends in opposite sides of  $R$  then  $S$  can be replaced by a cycle with the same length and bigger jump and with the required structure. Finally, if  $L(S_R) < \frac{1}{2}t$  then  $e(S) \leq \frac{2}{3} < e(R)$ , and so, we only have to prove the result for a path  $S$  of length bigger than  $\frac{3}{4}t$ .

The jump of  $S$  is the jump of  $S_R$  plus the jump of  $S_I$ . Then  $J(S) \leq L(S_R) + 2$ . The length of  $S$  is either  $L(S_R) + \lfloor \frac{t}{4} \rfloor$  or  $L(S_R) + \lceil \frac{t}{4} \rceil$ , so

$$e(S) \leq \frac{L(S_R) + 2}{L(S_R) + \lceil \frac{t}{4} \rceil} \leq 1 - \frac{\lfloor \frac{t}{4} \rfloor - 2}{3\lceil \frac{t}{4} \rceil} \leq \frac{2}{3} + \frac{8}{3t} < \frac{3}{4} \leq e(R)$$

□

**Corollary 8** *For  $q = 3$  and for any  $r \in \mathbb{Q} \cap [\frac{3}{4}, 1]$  exists a  $Q$ -state  $x$  with  $Es(x) = K$  such that  $e(x) = r$*

**PROOF.** Let  $r = \frac{a}{b}$  with  $a, b$  even integers. Has before, we construct a square



cycle  $P$  with  $j = 3K(q - p)$  times 0112 and  $i = K(4p - 3q)$  times 012 for any even  $K$ . The jump of  $P$  is  $J(P, x) = 3K(3(q - p) + (4p - 3q)) = 3Kp$  and its length is  $L(P) = 3K(4p - 3q) + 4K(3(q - p)) = 3Kq$ , so his efficiency is  $r$  and is a maximum efficiency cycle.  $\square$

## 4 Our Main Theorem

In this section we show how to construct a family  $\{y^n : n \geq 1\}$  of  $Q$ -state on  $K$  each having a polynomial number  $t_n$  of non-quiescent states and period  $T(y^n) = e^{\Omega(\sqrt{t_n})}$ .

The skeleton of  $y^n$  contains  $n$  cycles  $C_k$ , for  $k = 1, \dots, n$ , located in disjoint parts of  $K$  and such that their evolutions are independents. There is also a vertex  $v_0$  which has roughly the same distance in the skeleton of  $y^n$  to each cycle  $C_k$ , for  $k = 1, \dots, n$ . The evolution of  $v_0$  will depend upon all the evolutions of the cycles  $C_k$ ,  $k = 1, \dots, n$ , in a manner similar to those shown in Figure 1.

The construction of  $y^n$  is carried out as follows. We first construct, for each  $k = 1, \dots, n$ , a  $Q$ -state  $x$  in a finite rectangular grid  $R_k$  and show that the skeleton of  $R_k$  on  $x$  is stable. The cycle  $C_k$  will be the one on the boundary of  $R_k$ . Secondly we show how to embed  $x$  to a  $Q$ -state  $z^k$  of  $K$  such that the evolution of each vertex of  $K$  is the same as those appearing in the vertices of  $C_k$ . Thirdly, we construct  $y^n$  by merging all the  $Q$ -states  $z^k$ , for  $k = 1, \dots, n$  appropriately.

### 4.1 Construction of $R_k$

The construction of  $R_k$  and the  $Q$ -state  $x$  on  $R_k$  is based on the preliminary construction of a graph  $G_k$  and a  $Q$ -state  $w$  on  $G_k$ .

Let  $q$  be an even integer  $q \geq 4$  and let  $p$  be an even integer such that  $q \leq p \leq \lfloor \frac{3}{2}q \rfloor$ .

Let  $B_i$  be the rectangular grid of size  $i \times 2$  of  $\mathbb{Z}^2$  given by

$$B_i = \{(s, t) : s = 0, \dots, i - 1; t = 0, 1\}.$$

Let  $v \in \mathbb{Z}^2$  and  $B \subseteq \mathbb{Z}^2$ . We define

$$B + v = \{u + v : u \in B\}$$

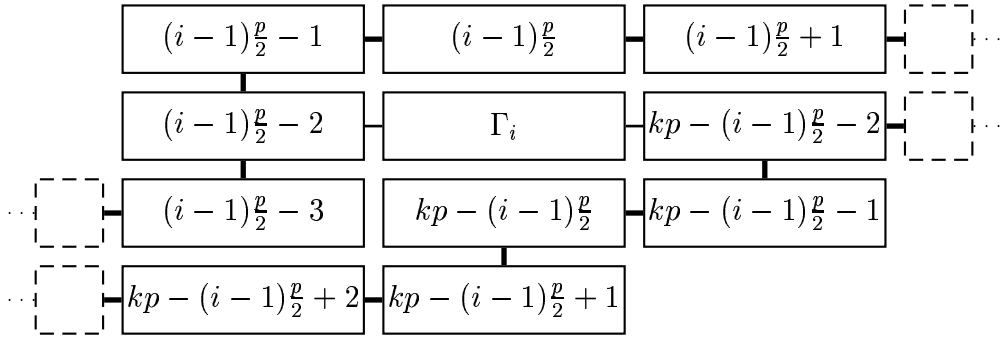


Fig. 2. Numbering of the cycle  $C_k$  and inner vertices.

where addition is taken in  $\mathbb{Z}^2$ .

The graph  $G_k$  is constructed iteratively as follows. Let  $G_1 := B_{\frac{p}{2}}$ . The graph  $G_i$  is the graph induced by  $(V(G_{i-1}) \setminus \{u_i\}) \cup (B_{\frac{p}{2}+1} + t_i)$ , where  $u_i = (i-1)(\frac{p}{2} - 3, 2) + (2, -2)$ ,  $t_i = u_i + (-2, 2)$  and  $V(G_{i-1})$  denotes the set of vertices of  $G_{i-1}$ . Then  $G_k$  can be decomposed in a cycle  $C_k = v_0, \dots, v_{kp-1}, v_{kp} = v_0$  which clockwise follows the boundary of  $G_k$  and the set of vertices  $\Gamma_i := t_i + (1, 0)$ ,  $i = 2, \dots, k$ . The vertex  $\Gamma_i$  is called an inner vertex of  $G_k$ , for  $i = 2, \dots, k$  (see Figure 2). Let us denote by  $S_k$  the spanning subgraph of  $G_k$  whose set of edges is the set of all edges of  $C_k$  and all horizontal edges incident with inner vertices of  $G_k$ .

In the sequel all the additions on indices are taken modulo  $kp$ . Let  $\mathcal{A}_k$  be the set of all sequences  $a = (a_i : i = 0, \dots, kp - 1)$  satisfying the following properties.

- (1) For every  $i = 0, \dots, kp - 1$ ,  $a_{i+1} \in \{a_i, s(a_i)\}$ .
- (2) For every  $i = 0, \dots, k\frac{p}{2} - 1$  the states  $a_i$  and  $a_{kp-1-i}$  are not adjacent in  $Q$ .
- (3) For every  $j = 0, \dots, k - 1$  and for every  $i = 0, \dots, kp - 1$ 
  - (a)  $a_{i+jp} \in \{s^{-1}(a_i), a_i, s(a_i)\}$ .
  - (b)  $a_{i+jp-3} \in \{s^{-2}(a_i), s^{-3}(a_i)\}$ .
  - (c)  $a_{i+jp+3} \in \{s^2(a_i), s^3(a_i)\}$

Notice that the cyclic rotation function  $\sigma : \mathcal{A}_k \rightarrow \mathcal{A}_k$  given by  $\sigma_i(a) = a_{i+1}$  is an isomorphism on  $\mathcal{A}_k$ .

Let  $a$  be in  $\mathcal{A}_k$ . We define a  $Q$ -state  $w(a)$  in  $G_k$  as follows. In  $C_k$  it assigns the state  $a_i$  to the vertex  $v_i$ , for  $i = 0, \dots, kp - 1$ . Let  $i \in \{2, \dots, k\}$  and let  $\alpha_v$  denote the state of the vertex  $\Gamma_i + v$ , for  $v \in \mathbb{Z}^2$  such that  $\Gamma_i + v \in G_k$ . Using Property 3a in the definition of  $\mathcal{A}_k$  it can be seen that  $\alpha_{-(1,1)}$  is adjacent with  $\alpha_{(2,0)}$  in  $Q$ . Hence, we assign to the vertex  $\Gamma_i$  the state  $\alpha_{-(1,1)}$  if  $\alpha_{-(1,1)} = s(\alpha_{(2,0)})$  or the state  $\alpha_{(2,0)}$  otherwise.

**Lemma 9**  $\forall k \geq 1$ ,  $Es(G_k, w(a)) = S_k$ , for every  $a \in \mathcal{A}_k$ .

**PROOF.** From our assignment of states to the inner vertices of  $G_k$  and since  $a_{i+1} \in \{a_i, s(a_i)\}$ , for  $i = 0, \dots, kp - 1$ , we have that the subgraph  $S_k$  is contained in the skeleton of  $G_k$ .

Using the Property 2 in the definition of  $\mathcal{A}_k$  we deduce that the set of all vertical edges in the skeleton of  $w(a)$  on  $G_k$  is the set formed by all vertical edges in  $C_k$ .

Using the Property 3c we deduce that the horizontal edges of the skeleton not adjacent to any inner vertex are exactly the horizontal edges of  $C_k$ .

It remains to show that  $\Gamma_i$  is adjacent in the skeleton with neither  $\Gamma_i + (0, 1)$  nor  $\Gamma_i + (0, -1)$ .

From Property 3c we know that  $\alpha_{(0,1)} \in \{s^2(\alpha_{-(1,1)}), s^3(\alpha_{-(1,1)})\}$  and from Property 3b that  $\alpha_{(2,0)} \in \{s^{-2}(\alpha_{(0,1)}), s^{-3}(\alpha_{(0,1)})\}$ . Let us assume that  $\alpha_{(0,1)} = s^2(\alpha_{-(1,1)})$ . Then  $\alpha_{(2,0)} \in \{\alpha_{-(1,1)}, s^{-1}(\alpha_{-(1,1)})\}$ . From the definition of the state of  $\Gamma_i$  we deduce that  $\alpha_{(0,0)} = \alpha_{-(1,1)}$ . From Property 3c we finally get  $\alpha_{-(0,1)} \in \{s^2(\alpha_{-(1,1)}), s^3(\alpha_{-(1,1)})\}$ . Therefore, the state  $\alpha_{(0,0)}$  is adjacent with neither  $\alpha_{-(0,1)}$  nor  $\alpha_{(0,1)}$ . The case when  $\alpha_{(0,1)} = s^3(\alpha_{-(1,1)})$  can be handle in a similar way.  $\square$

**Theorem 10** *For every  $k \geq 1$  and for every  $a \in \mathcal{A}_k$  we have that*

$$F(w(a)) = w(\sigma(a))$$

**PROOF.** Let  $v$  be a vertex in the cycle  $C_k$  not adjacent to any inner vertex in the skeleton  $S_k$ . Then  $F_v(w(a)) = w_u(a)$  where  $u$  is the next clockwise neighbor of  $v$ . Then  $w_u(a) = w_v(\sigma(a))$ .

We prove the property for a vertex  $v$  in  $C_k$  having some inner vertex as neighbor in  $S_k$ . In this situation we know that the vertex  $v$  is either  $\Gamma_i - (1, 0)$  or  $\Gamma_i + (1, 0)$ , for some  $i = 2, \dots, k$ . Let us assume that  $v = \Gamma_i - (1, 0)$ . From Property 1 we know that  $\alpha_{(-1,1)} \in \{s(\alpha_{-(1,1)}), s^2(\alpha_{-(1,1)})\}$ . Hence, if  $\alpha_{(0,0)} = s(\alpha_{-(1,1)})$  then  $F_{\Gamma_i - (1,0)}(w(a)) = w_{\Gamma_i + (-1,1)}(a) = w_{\Gamma_i - (1,0)}(\sigma(a))$ . In the case that  $\alpha_{(0,0)} = \alpha_{-(1,1)}$  we know from Property 1 that  $\alpha_{(-1,0)} \in \{\alpha_{(0,0)}, s(\alpha_{(0,0)})\}$  and then  $F_{\Gamma_i - (1,0)}(w(a)) = w_{\Gamma_i + (-1,1)}(a) = w_{\Gamma_i - (1,0)}(\sigma(a))$ . The property for the vertex  $\Gamma_i + (1, 0)$  can be proved similarly.

Finally, doing a case analysis it is possible to show that  $F_{\Gamma_i}(w(a)) = \beta_{(-1,1)}$  if  $\beta_{(-1,1)} = s(\beta_{(2,0)})$  and that it is  $\beta_{(2,0)}$  otherwise, where  $\beta_v$  denotes the state of the vertex  $\Gamma_i + v$  in  $F(w(a))$ . Then  $F_{\Gamma_i}(w(a))$  is the state associated to  $\Gamma_i$  by  $w(\sigma(a))$ .  $\square$

**Corollary 11** *For every  $k \geq 1$  and for every  $a \in \mathcal{A}_k$ ,  $Es(G_k, F(w(a))) = S_k$ .*

Previous Corollary says that the skeleton of  $w(a)$  on  $G_k$  is stable. It is not hard to see that the maximum efficiency of  $w(a)$  is  $\frac{p}{q}$ . Moreover, the cycle  $C_k$  is the unique cycle with this efficiency.

Let  $R_k$  be the smallest rectangular grid on  $\mathbb{Z}^2$  containing  $G_k$ . Then  $R_k$  is a  $k(\frac{p}{2} - 2) + 2 \times 2k$  rectangular grid. We now describe how to obtain the  $Q$ -state  $x(a)$  on  $R_k$  from the  $Q$ -state  $w(a)$  on  $G_k$ . Let  $v \in V(R_k) \setminus V(G_k)$  be with exactly two neighbors  $u$  and  $w$  in  $V(G_k)$  and such that there exists a vertex  $v'$  in  $G_k$  adjacent to  $u$  and  $w$ . We assign to  $v$  the state of  $v'$ . We obtain  $x(a)$  by iterating this process until a state be given to all vertices of  $R_k$ . Clearly, the maximum efficiency of  $x(a)$  on  $R_k$  is  $\frac{p}{q}$  and  $F(x(a)) = x(\sigma(a))$ . Moreover, the sequence of states of  $x(a)$  appearing in the boundary of  $R_k$  when we follow it clockwise is  $\sigma^j(a)$ , for some  $j = 0, \dots, kp - 1$ .

#### 4.2 Construction of $z^k$

The  $Q$ -state  $x(a)$  on  $R_k$  has a canonical extension to  $K$  (still denoted  $x(a)$ ) which assigns the state 0 to all the vertices outside  $R_k$  (seen as a subgraph of  $K$ ). It is not hard to see that the skeleton of  $x(a)$  on  $K$  is not stable. Even worth if we iterate  $x(a)$  with the NCN rule  $F$  the skeleton will increases and a cycle of greater efficiency than  $\frac{p}{q}$  will appear.

In order to obtain  $z^k = z(a)$  we proceed to modify a finite number of states of  $x(a)$  outside  $R_k$ . A vertex  $v$  in the boundary of  $R_k$  is called a *starting* vertex if its state is 1 and it has a neighbor in the boundary of  $R_k$  in the state 0. Let us denote by  $v'_0, \dots, v'_{kp-1}$  be the vertices appearing in the boundary of  $R_k$  when we follow it counter-clockwise. Then  $v'_i = v'_{kp-1-i}$ , for all  $i = 0, \dots, kp - 1$ . Let  $b_i$  be the state of the vertex  $v'_i$  in  $x(a)$ . We associate to the vertex  $v := v'_i$  the sequence  $b^v$  given by  $b_1^v = b_{i+1}, b_2^v = b_{i+2}, \dots, b_t^v = b_{i+t}$  where  $v'_{i+t}$  is a starting vertex and no vertex  $v'_{i+1}, \dots, v'_{i+t-1}$  is a starting vertex (still all additions are taken modulo  $kp$ ). We denote by  $l(u)$  the length of the sequence  $b^u$ . We associate the empty sequence to all the remaining vertices (those not in the boundary of  $R_k$ ). The following procedure will modify the sequence  $b^v$  in some vertices of  $K$ . Let  $S$  be the set of all starting vertices.

#### Associating-sequences

```

begin
   $W := S$ ;
  while (  $W \neq \emptyset$  ) do.
    Let  $v := v'_i \in W$ ;
     $u := v'_{i-1}$ ;
     $W := W \setminus \{v\}$ .

```

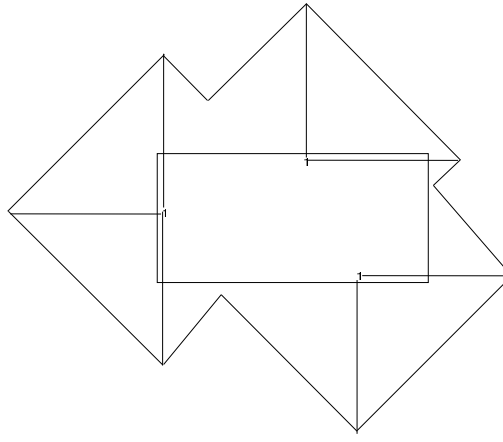


Fig. 3. Shape of the set  $D(z(a))$ . The inner rectangle represents the graph  $G_k$ .

```

if  $b_2^v$  and  $x(a)_u$  are not neighbors in  $Q$ 
then
     $b^u := b_2^v, \dots, b_{l(v)}^v$ .
     $W := W \cup \{u\}$ .
else
     $b^u := b_4^v, b_3^v, b_4^v, b_5^v, \dots, b_{l(v)}^v$ .
endwhile
end

```

Let  $(\bar{b}^u)_{u \in K}$  be sequences obtained in all the vertices after applying the previous algorithm. Notice that only finitely many vertices have a non-empty sequence  $\bar{b}^u$ .

We say that we *complete* a vertex  $u$  in the direction  $d$  (with the sequence  $\bar{b}^u$ ) if for all  $i = 1, \dots, l(u)$  we assign the state  $\bar{b}_i^u$  to the vertex  $u + id$ . Let  $v$  be a vertex in the boundary of  $R_k$ . A direction  $d$  is *outgoing* for  $v$  if  $v + d$  is a neighbor of  $v$  (in  $K$ ) not in  $R_k$ . In order to obtain the  $Q$ -state  $z(a)$  we first *complete* all the vertices in their outgoing directions. Secondly, for each corner  $v$  of  $R_k$  with outgoing directions  $d_1$  and  $d_2$  we assign the state  $\bar{b}_i^v$  to all the vertices (outside  $R_k$ )  $v + i_1 d_1 + i_2 d_2$  with  $i_1, i_2 \in \mathbb{Z}$ ,  $i_1, i_2 \geq 0$  and  $i_1 + i_2 = i \forall i = 1 \dots l(v)$ .

For a  $Q$ -state  $x$  we denote by  $C(x)$  the vertex set of the (only) infinite connected component of the subgraph induced in  $K$  by  $\mathbb{Z}^2 \setminus \text{supp}(x)$  and let us denote by  $D(x) := \mathbb{Z}^2 \setminus C(x)$ .

From the construction of  $z(a)$  it is clear that  $\forall v \in D(z(a))$ ,  $F_v(z(a)) = z_v(\sigma(a))$  (see Figure 3)

From Proposition 1 we know that the evolution of every  $v \notin D(z(a))$  is com-

pletely determined by the states in  $D(z(a))$ . More precisely we have.

**Corollary 12** *Let  $k \geq 1$  and  $a \in \mathbb{A}_k$  be. Then*

- $F_v(z(a)) = z_v(\sigma(a))$  for all  $v \in D(z(a))$ .
- $F_v^t(z(a)) = 0$ , for all  $v \notin D(z(a))$  and  $t = 0 \dots, d - 1$ .
- $F_v^t(z(a)) = F_u^{t-d}(z(a))$ , for all  $v \notin D(z(a))$  and  $t \geq d$ .

where  $d$  is the (smallest) distance from  $v$  to  $D(z(a))$  in  $K$  and  $u$  is any vertex in  $D(z(a))$  to distance  $d$  of  $v$ .

### 4.3 Final result

In order to prove our main theorem let  $n$  be any integer  $n \geq 1$ . Let  $k_1, \dots, k_n$  be  $n$  integers and let  $a^1, a^2, \dots, a^n$  be  $n$  sequences such that for all  $i = 1, \dots, n$  the sequence  $a^i$  belongs to  $\mathcal{A}_{k_i}$ .

The addition of two  $Q$ -states  $x$  and  $y$  on  $K$  with disjoint supports is the  $Q$ -state  $z$  defined by  $z_v = x_v$  if  $y_v = 0$  and  $z_v = y_v$  otherwise. This definition can be extended to  $n$   $Q$ -states with disjoint supports in the obvious manner.

Let  $x$  be a  $Q$  state on  $K$  and  $u \in \mathbb{Z}^2$ . We define the shift of  $x$  on  $u$  as the  $Q$ -state  $M^u(x)$  given by  $M_v^u(x) = x_{u+v}$ , for all  $v \in \mathbb{Z}^2$ . Then  $M^0(x) = x$ .

Let  $d$  be large enough such that there are  $u_1, \dots, u_n$  with  $\bar{z}^i := M^{u_i}(z(a^i))$ , for  $i = 1, \dots, n$  satisfy the following properties.

- The  $Q$ -states  $\bar{z}^i$ , for  $i = 1, \dots, n$  have disjoint support.
- The distance from  $(0, 0)$  to  $D(\bar{z}^i)$  is  $d$ , for all  $i = 1, \dots, n$ .

Let  $y^n$  be the  $Q$ -state obtained as the addition of all the  $Q$ -states  $\bar{z}^i$ ,  $i = 1, \dots, n$ . Clearly, the efficiency of  $y^n$  is  $\frac{q}{p}$  since each  $\bar{z}^i$  has this efficiency, for  $i = 1, \dots, n$ . Moreover, its skeleton is stable.

**Theorem 13** *For every  $q \geq 4$  and every even integer  $p$  such that  $q \leq p \leq \lfloor \frac{3}{2}q \rfloor$  and, for every integer  $n \geq 1$  and integers  $k_1, k_2, \dots, k_n$  there exists a  $Q$ -state  $y^n$  of  $K$  such that  $T_{(0,0)}(y^n) = p \cdot \text{lcm}_{i=1, \dots, n} \{k_i\}$ .*

**PROOF.** Given  $n$  and  $k_1, \dots, k_n$  let  $a^1, a^2, \dots, a^n$  be  $n$  sequences such that for all  $i = 1, \dots, n$  the sequence  $a^i$  belongs to  $\mathcal{A}_{k_i}$ .

We define  $y^n$  as described above. Since the skeleton of  $y^n$  is stable we know from Proposition 1 that the evolution of the vertex  $(0, 0)$  depends only on the evolution of a vertex  $v_i \in D(z(a^i))$  with distance  $d$  to it.

Our goal now is to choose appropriated sequences  $a^i$ ,  $i = 1, \dots, n$  so as the period in vertex  $(0, 0)$  is  $p \cdot \text{lcm}_{i=1, \dots, n} \{k_i\}$ .

Let  $a'$  be any sequence in  $\mathcal{A}_1$  that starts as follows  $(1, 2, 2, 3, 4, \dots, q - 1, 0)$  and let  $a''$  be the sequence obtained from  $a'$  by the following modification  $(1, 1, 2, 3, \dots, q - 1, 0)$ . The sequences  $a^i$  is composed by  $k_i - 1$  times the subsequences  $a'$  followed by the subsequence  $a''$ , that is

$$a^i = (1, 2, 2, 3, \dots, 0)^{(k_i-1)}(1, 1, 2, 3, \dots, 0)$$

Then the sequence of states of the vertex  $(0, 0)$  is given by

$$(1, 2, 2, \dots, 0)^{\kappa-1}(1, 1, 2, \dots, 0)$$

where  $\kappa = \text{lcm}_{i=1, \dots, m} \{k_i\}$ . Therefore  $T_v(y) = p \cdot \kappa$ .  $\square$

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