Integrated traffic-transit stochastic equilibrium model including park and ride facilities☆

Cristobal Pineda[a,*], Cristián E. Cortés[a], Pedro Jara-Moroni[b], Eduardo Moreno[c]

[a] *Department of Civil Engineering, Transport Division, Universidad de Chile*
[b] *Department of Economics, Universidad de Santiago de Chile*
[c] *Faculty of Engineering and Sciences, Universidad Adolfo Ibañez*

Abstract

We propose an integrated stochastic equilibrium model considering private car traffic as well as transit networks, where the interaction appears at two levels, namely total travel times as well as modal share. The assignment for traffic equilibrium is based on the MTE model of Baillon and Cominetti (2008), while the equilibrium on the transit network is represented by the STE model by Cortés et al. (2013). Stochastic travel decisions are made at a node level, avoiding enumeration of routes or strategies and incorporating different perceptions and uncertainty issues. In the general version of our model, travelers are allowed to change from car to transit at specific locations in a park and ride scheme. We propose an MSA algorithm to calculate a stochastic integrated equilibrium and conduct numerical experiments on real networks that highlight the effect of stochasticity on equilibrium flows and travel times. Our experiments show that higher stochasticity implies more dispersion on equilibrium flows and longer expected travel times.

Keywords: Transit Equilibrium, Traffic Equilibrium, Stochastic Models, Congested networks, Park and ride

1. Introduction

In the last decades, many cities over the world have grown considerably in terms of both population and land deployment, generating new necessities

☆The authors thank the financial support of Fondecyt Chile Grant 1100239 and 1130681, the Millennium Institute "Complex Engineering Systems" (ICM: P-05-004-F, CONICYT: 522 FBO16) and the FONDEF project D10I1002.

*Corresponding author

Email addresses: `crpineda@ing.uchile.cl` (Cristobal Pineda), `ccortes@ing.uchile.cl` (Cristián E. Cortés), `pedro.jara@usach.cl` (Pedro Jara-Moroni), `eduardo.moreno@uai.cl` (Eduardo Moreno)

and requirements of transportation provision for the inhabitants of the different cities, most of them affected by serious traffic congestion during rush hours; this observed fact has motivated the interest of researchers and practitioners in modeling urban networks at different scales for various purposes. In this context, the use of urban planning models to assess investment policies for improving the welfare of people has become an important issue. At a strategic level of analysis, the so-called assignment and equilibrium users models are designed to reproduce the observed behavior and choices of individuals with regard to transit and traffic networks; on the one hand, transit equilibrium models seek to reproduce the boarding and alighting stops and routes choices, in terms of transit lines used, while on the other, traffic equilibrium models seek to reproduce the routes choices over the urban road network.

There is abundant literature on assignment and equilibrium models exploring pure modes, mainly for the case of traffic networks, which generally relies on the Wardrop's principle, indicating that rational users choose routes that minimize their expected travel time. In turn, many of the existing transit equilibrium models have adopted this principle. However, between traffic and transit modes there is an irremediable difference: while in the case of traffic, choosing the route that minimizes the on-board expected travel time is enough, in the transit dimension the route choice is defined by the particular bus that a passenger boards, within a set of different common lines that serve a bus-stop that can be used to reach the destination. Then, in addition to considering the on-board travel time on the vehicle, in the transit dimension the waiting time also plays an important role, which is linked to other variables inherent to any transit system, such as frequency and bus capacity.

In terms of transit users behavior, most recent literature has been aimed at modeling the preferences assuming that passengers choose a route *strategy* for their trips. Originally, Spiess and Florian (1989) define a *strategy* as a set of rules that, when applied, allow a passenger to reach his(her) destination, and decisions are made at each intermediate node where that passenger boards a bus. A well-formulated strategy includes the choice of the attractive lines set at a stop, as described by Chriqui and Robillard (1975), and besides, this notion implies that users have a completely knowledge of the network structure and the conditions for recognizing and using effective strategies (Bouzaïene-Ayari et al., 2001). The problem of minimizing the expected generalized cost of transit passengers can be modeled as a user equilibrium in the *hyperpaths* space, a concept introduced by Nguyen and Pallotino (1988). An *hyperpath* is an acyclic graph that connects an origin to a destination, and reflects a strategy to be followed by a particular user. This description can assimilate the passengers assignment problem on a transit network to a standard vehicle assignment problem in the traffic dimension.

Early studies in this area underestimate a relevant aspect present in the passengers assignment process, which is the congestion that occurs in stops when the bus capacity is not enough to serve the whole demand. The first models (Nguyen and Pallotino, 1988; Spiess and Florian, 1989) work reasonably well under low demand scenarios at bus stops; if this is not the case, there may be passengers unable to board the first bus that belong to the set of attractive lines, which obviously increases their waiting times. De Cea and Fernández (1993) incorporate congestion at bus stops, assuming that passengers travel through a sequence of intermediate nodes, allowing choices within a set of common lines at a stop only if they all share the next stop to be served. However, the set of common lines is computed through a heuristic approach, and therefore equilibrium conditions are not guaranteed. Cominetti and Correa (2001) develop a transit equilibrium model based on hyperpaths that explicitly includes the effects of congestion by means of a queuing model at bus stops. In addition, the general equilibrium model allows multiple origins and destinations, transit lines overlapped in certain routes segments, and transfers at intermediate nodes. The resolution of the model is performed using a dynamic programming approach and the authors prove the existence of a user equilibrium in the network. Cepeda et al. (2006) extend the previous model, obtaining a new characterization of equilibrium by formulating an optimization problem, which leads to a gap function that vanishes when equilibrium is reached. These models consider that passengers always choose the optimal strategy or hyperpath that minimizes the generalized travel costs, leading to a deterministic approach.

Cortés et al. (2013) extend the latter model, proposing a stochastic approach that considers both congestion at bus stops and a stochastic behavior of passengers during the boarding process. This model is denoted *Stochastic transit equilibrium* (STE) and reflects the perception of passengers related to the level of service of a specific line, including facts like traffic conditions of the transit network, reliability of the line, etc. The model includes stochasticity by means of a probability distribution associated with boarding a bus belonging to a specific line, which can be characterized by the frequency observed in a given stop along with the expected travel time to the next stop. That formulation generates a stochastic common lines problem, in which each line has a probability of being chosen by a passenger, even if the quality of service is poor. In addition, the formulation incorporates capacity constraints at stops. A significant difference between this formulation and similar deterministic approaches in the literature is that it is no longer necessary to enlist all the feasible strategies. This is possible because the expected travel time values can be analytically computed for a given destination together with the equilibrium flows on each line, solving a simultaneously set of common lines problems, interrelated by flow conservation constraints at each node. Moreover, Cortés et al. (2013) propose an algorithm to find the stochastic equilibrium.

In the field of vehicle assignment on traffic networks, the trend has been in the development of stochastic models, considering variability among users to perceive the travel costs. Dial (1971) was a pioneer in formulating a stochastic assignment model that excludes the congestion effects, in which the demand of each origin-destination pair is distributed using a discrete route-based logit choice model. This model was extended by Fisk (1980) for flow-dependent costs, which leads to an equivalent formulation of an optimization problem. The assumption behind such models is that the error terms in routing costs are independent Gumbel random variables. However, this assumption is highly unlikely when there are overlapping routes, and therefore Daganzo and Sheffi (1977) proposed an alternative model based on a probit formulation for the stochastic assignments, computed by simulations.

The previous models for stochastic traffic assignment are mostly route-based models. Hence, it is assumed that users *a priori* choose an optimal route from their origins; this approach involves several problems: first, in models based on either logit or probit assignments, there always exist a probability of choosing a route, even if that route has a very high cost relative to the others. Second, these models assume independent paths, even if they have overlapping segments, existing an obvious correlation. And finally, these models require paths enumeration, which can become computationally impractical for large enough networks. To address these difficulties, Baillon and Cominetti (2008) propose a stochastic traffic equilibrium model based on discrete choice under a sequential arcs selection process at each intermediate node, instead of basing the decision on the entire route. This process is governed by an embedded Markov chain, and so the authors called the model *Markovian traffic equilibrium* (MTE). They prove that this formulation leads to a strictly convex minimization problem which avoids paths enumeration, proposing also computational methods that are effective even for large networks.

In multimodal user equilibrium models, significant advances over the past four decades have been observed (Florian, 1977; Florian and Spiess, 1983; Wong, 1998). These models assume that passengers will only choose pure modes to perform the entire trip, using a logit formulation to obtain the proportion of trips in each mode, according to generalized costs of each of the alternatives. However, in the last 20 years, it has been a trend on transportation policies to improve public transportation attractiveness by decreasing the volume of cars moving on streets and encouraging modal interchange. Hence, *park and ride* facilities have emerged in specific locations of urban zones, for facilitating a first leg of the trip conducted using a personal car, followed by a second leg completing the trip through a massive and efficient mode of public transport, namely train, bus, or subway. These trips are performed by a not unique transport mode known in the literature as *combined* modes. The incentive for users to choose these combined modes is associated with the congestion on the streets, frequency and fares of transit services, and the location of parking facilities. User equilibrium involving networks with combined modes is a line of research with only few major developments. In fact, existing models taking this fact into account are exactly two: Florian and Los (1979) develop a model that determines the origin-destination matrices from the origin to the parking lot. The goal is to predict changes in flow depending on the policies adopted that could involve implementation of parking lots, such as capacity or parking fares. Fernández et al. (1994) present some approaches to formulate a user equilibrium including combined modes, modeling the choice of transfer nodes through a nested logit model; this model assumes symmetric cost functions, which reduces the range of applications of the model. To address this, García and Marín (2005) extended the previous model to include asymmetric cost formulations, using nested logit formulation in two steps: first, describing the mode choice by the user; and later, the choice of the transfer node. The authors assume a deterministic user principle governing the route choice in each mode.

The goal of the present paper is to develop an integrated stochastic equilibrium model considering both traffic and transit networks, to incorporate the interactions between the two pure modes in terms of travel time and generalized costs. In addition, our model adds the combined mode option into the analysis. The integrated formulation puts together the MTE model developed by Baillon and Cominetti (2008) for the traffic network, and the STE model of Cortés et al. (2013) for the transit network. Both models share similarities in their formulation, as travel decisions are made in both cases at the node level, avoiding enumeration of routes or strategies. Moreover, both approaches include the effect of congestion, at vehicular and passenger levels, and both include stochasticity as a central feature of each model, allowing to incorporate the different perceptions and uncertainty issues that people have about the features and conditions of the urban network, to better reflect what really occurs in large urban centers. To make this proposal realistic and applicable in real modeling conditions, we also propose an algorithm that performs the resulting stochastic equilibrium over a generic traffic and transit network, which is tested with real data of a large city, using on the transit side a logistic function for the boarding probability at bus stops, and on the traffic one a Gumbel distribution for the error term in expected travel times. In this paper, apart from developing the integration details, we apply the algorithm to a real case scenario corresponding to a medium size network of the city of Iquique-Chile.

The ultimate purpose of the integrated equilibrium model is to become an urban planning tool for transportation decision makers. The model has to be calibrated in a case by case modality, to be able to reproduce the current passengers and vehicles flows observed on the streets. Then, such tool can become a powerful prediction model of users behavior when relevant changes in public transportation or road infrastructure supply are implemented. The paper is organized as follows, in the next section the foundation of the MTE as well as the STE models are presented. Next, the integrated formulation is presented, to follow in section 4 with the summary of the solution algorithms, along with many experiments to show the potential and consistency of the stochastic formulation under different scenarios. The paper close with a summary, conclusions and ideas for further work.

2. Preliminaries

2.1. Stochastic transit equilibrium model

The notion of stochastic transit equilibrium is developed by Cortés et al. (2013), extending the deterministic formulation of Cominetti and Correa (2001) and Cepeda et al. (2006), based on minimum hyperpath choice.

Consider a directed graph $G = (N, A)$, and denote by i_a and j_a the tail and head nodes of an arc $a \in A$. Let $A_i^+ = \{a \in A : i_a = i\}$ and $A_i^- = \{a \in A : j_a = i\}$ be the sets of outgoing and incoming arcs from/to node $i \in N$ respectively. Let $d \in D \subseteq N$ be the subset of destination nodes within the network. For each $d \in D$ and every node $i \neq d$ a fixed demand $g_{id} \geq 0$ is given. To keep the model tractable we need to specify arc-destination flows. The set $\mathcal{V} := \mathbb{R}_+^{|A|\times|D|}$ denotes the space of arc-destination flow vectors v with nonnegative entries $v_{ad} \geq 0$, while $\mathcal{V}_0$ is the set of feasible flows $v \in \mathcal{V}$ such that $v_{ad} = 0$ for all $a \in A_d^+$ (i.e. no flow with destination d exits from d) and satisfying the flow conservation constraints:

$$g_{id} + \sum_{a \in A_i^-} v_{ad} = \sum_{a \in A_i^+} v_{ad} \qquad \forall i \neq d. \tag{1}$$

Let $v_a = \sum_{d \in D} v_{ad}$ be the total flow on link a. Each link $a \in A_i^+$ is associated with a line that stops at node i, and is characterized by a continuous travel time function $t_a : \mathcal{V} \to [0, \bar{t}_a[$, where $\bar{t}_a$ is a finite upper bound, and the effective frequency function $f_a : \mathcal{V} \to [0, +\infty]$ which is either identically $+\infty$ or everywhere finite, in which case, for each $d \in D$ we assume that $f_a \to 0$ when $v_a \to \bar{v}_a$ with $f_a(v)$ strictly decreasing with respect to v_a when strictly positive. These functions reflect the congestion of transit lines. In particular, when a line is completely congested, the observed frequency of that line by a passenger waiting at that stop is zero.

Consider a passenger traveling to destination d that reaches an intermediate node i on his or her trip, as shown on Fig. 1.

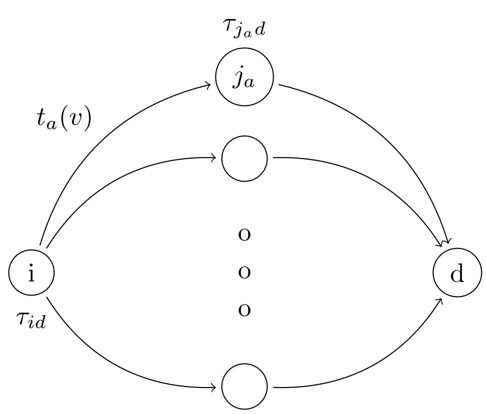


Figure 1: Common lines problem on a general transit network

To exit from i, the passenger can choose one of the arcs $a \in A_i^+$ to reach the next node j_a. Call τ_{id} the expected total travel time from node i to destination d. In the common-line problem, the passenger compares the times $t_a(v) + \tau_{j_a d}$ to choose the arcs to follow. In the stochastic common-line problem, each passenger has probability p_a^d of wishing to board a bus of line a to reach destination d, given that a bus of line a is at the bus stop:

$$p_a^d = \mathbb{P}(\text{boarding a bus to reach destination } d | \text{bus of line } a \text{ is at the stop}).$$

A passenger that wishes to travel from i to d compares the expected travel time of boarding the current bus, $t_a(v) + \tau_{j_a d}$, with the expected travel time of waiting for the next bus, τ_{id}. This probability is given by a stochastic model and depends on the expected travel time. We will assume that p_a^d is a strictly decreasing continuous function of the difference between expected travel time on the current bus $t_a(v) + \tau_{j_a d}$ and expected travel time of waiting τ_{id}, $\varphi_a : \mathbb{R} \to]0, 1[$:

$$p_a^d \equiv \varphi_a(t_a(v) + \tau_{j_a d} - \tau_{id}) \,. \tag{2}$$

We introduce these functions merely as tools that are used to model the stochastic decision of boarding and are to be determined by the modeler.

Cortés et al. (2013) assume that the arrival process is completely renewed each time a bus arrive to a certain stop, and that the arrival of buses follow a Poisson distribution. These assumptions allow us to calculate expected travel time and flow assignment (respectively) as in equations (4) and (5) below.

The stochastic transit equilibrium is formulated as a set of simultaneous stochastic common-lines problems (one for each id pair), coupled by flow conservation constraints (Cortés et al., 2013). We define for each $v \in \mathcal{V}$ the *flow entering node i with destination d* by:

$$x_{id}(v) := g_{id} + \sum_{a \in A_i^-} v_{ad}. \tag{3}$$

Definition 1. *A pair of feasible flow vector and expected travel times $(v^*, \tau^*) \in \mathcal{V}_0 \times \mathbb{R}_+^{|N| \times |D|}$ is a Stochastic Transit Equilibrium if for all $d \in D$ and $i \in N$,* *with $i \neq d$ we have:*

$$\tau_{id}^* = \frac{1 + \sum_{a\in A_i^+} p_a^d f_a(v^*) \left(t_a(v^*) + \tau_{j_a d}^*\right)}{\sum_{a\in A_i^+} f_a(v^*)\, p_a^d} \qquad ; \tag{4}$$

$$v_{ad}^* = x_{id}(v^*)\, \frac{f_a(v^*)\, p_a^d}{\sum_{a\in A_i^+} f_a(v^*)\, p_a^d} \quad , \qquad \forall a \in A_i^+; \tag{5}$$

$$p_a^d = \varphi_a(t_a(v) + \tau_{j_a d} - \tau_{id}) \quad , \qquad \forall a \in A_i^+.$$

2.2. Markovian traffic equilibrium model

The basis of Markovian equilibrium model (MTE) in traffic networks of Baillon and Cominetti (2008) is that car users travel to their destinations by a sequential arc selection process based on a discrete choice model at every intermediate node i that they reach on their trip.

Let the random variable $\tilde{t}_a = t_a + \nu_a$ be the travel time on arc $a \in A_i^+$, where t_a is the deterministic travel time on that arc, and ν_a an error term, which represents variability between drivers perceptions. It is assumed that these terms have a continuous distribution, where $\mathbb{E}(\nu_a) = 0$. Then, the optimal travel time between all available paths $r \in R_{id}$ from i to d is given by:

$$\tilde{\tau}_{id} = \min_{r\in R_{id}} \left\{ \sum_{a\in r} \tilde{t}_a \right\}.$$

Given a destination d, a driver arrives at node i and compares the travel time or generalized costs using each of the outgoing arcs of the node i. Let $\tilde{z}_{ad}$ the stochastic time or cost to destination d using the arc a, which is:

$$\tilde{z}_{ad} = \tilde{t}_a + \tilde{\tau}_{j_a d} = z_{ad} + \epsilon_{ad} \tag{6}$$

where $z_{ad} = \mathbb{E}(\tilde{z}_{ad})$ and $\mathbb{E}(\epsilon_d) = 0$. The driver selects the arc having the shortest time between the set $a \in A_i^+$, according to their own perception. This process is repeated at each intermediate node during the trip. Then, for each destination $d \in D$, there is an underlying Markov chain in the network, where for a node $i \neq d$, the transition probabilities are given by:

$$p_a^d = \begin{cases} \mathbb{P}(\tilde{z}_{ad} \leq \tilde{z}_{a'd}, \forall a' \in A_{i_a}^+) & \text{if } a = (i_a, j_a) \in A_{i_a}^+ \\ 0 & \text{if not} \end{cases}$$

while the destination node is an absorbing state of the chain, i.e. $p^d_{(d,d)} = 1$.

The probability of using the arc a from node i to reach destination d can be expressed as:

$$\mathbb{P}(\tilde{z}_{ad} \leq \tilde{z}_{a'd} \ \ \forall a' \in A_i^+) = \frac{\partial \varphi_{id}}{\partial z_{ad}}(z_d) \tag{7}$$

where, for each pair id, $\varphi_{id} : \mathbb{R}^{|A_i^+|} \to \mathbb{R}$ is the expected travel time function:

$$\varphi_{id}(z_d) \equiv \mathbb{E}\left(\min_{a \in A_i^+} \{z_{ad} + \epsilon_{ad}\} \right). \tag{8}$$

The functions φ_{id}, which are component-wise non-decreasing, concave and smooth, are determined by the random variables ϵ_d, and in turn, by the variables ν_a. The functions that belong to this class, denoted by $\mathcal{E}$, and where $\varphi_{dd} \equiv 0$, admit an analytical characterization, as will be seen later on this section.

With these elements, it is possible to describe the Bellman's dynamic programming equations as $\tilde{\tau}_{id} = \min_{a \in A_i^+} \tilde{z}_{ad}$. Taking expectation at both sides of the equation, we have:

$$\begin{cases} z_{ad} = t_a + \tau_{j_a d} \\ \tau_{id} = \varphi_{id}(z_d) \end{cases}$$

which may be expressed only in terms of τ_{id} variables:

$$\tau_{id} = \varphi_{id}\left((t_a + \tau_{j_a d})_{a \in A_i^+} \right). \tag{9}$$

Furthermore, using the same notation as in the STE model, we have the same flow conservation constraints at each node as in Equation (1). Hence, the flow distribution can be written as:

$$\begin{cases} v_{ad} = x_{id} \cdot p_a^d \\ x_{id} = g_{id} + \sum_{a \in A_i^-} v_{ad} \end{cases}. \tag{10}$$

Given a family of functions $\varphi_{id} \in \mathcal{E}$ (one for each $id \in N \times D$ pair) with $\varphi_{dd} \equiv 0$, and, for each $a \in A$ a strictly increasing continuous travel time function $s_a : \mathbb{R} \to (0, \infty)$, the MTE model is formalized as follows:

Definition 2. *A vector $v \in \mathbb{R}^{|A|}$ is a Markovian traffic equilibrium iff $v_a = \sum_{d \in D} v_{ad}$ where the v_{ad}'s satisfy the flow distribution equation (10) with τ_{id}'s solving (9) for $t_a = s_a(v_a)$.*

Baillon and Cominetti (2008) used for large networks tests a BPR-type formulation for travel times functions $s_a(v_a)$ in order to include the vehicles congestion effect and a logit formulation family for $\varphi_{id}(\cdot)$, of parameters β_{id}. The latter leads to:

$$\tau_{id} = -\frac{1}{\beta_{id}} \ln \left(\sum_{a \in A_i^+} e^{-\beta_{id}(t_a + \tau_{j_a d})} \right), \tag{11}$$

$$p_a^d = \frac{e^{-\beta_{id}(t_a + \tau_{j_a d})}}{\sum_{a' \in A_{i_a}^+} e^{-\beta_{id}(\tau_{a'} + \tau_{j_{a'} d})}}. \tag{12}$$

An important issue of MTE is to ensure existence in the model, requiring that drivers reach their destinations in a finite time. That is, within the model, expected travel times can not be negative, because that would imply that certain vehicle flows were kept captive within certain cycles. Therefore, Cominetti et al. (2012) indicate that we can avoid the latter if the travel time vector t belongs to the set:

$$\mathcal{C} = \left\{ t \in \mathbb{R}^{|A|} : \exists \hat{\tau} \text{ with } \hat{\tau}_{id} < \varphi_{id} \left((t_a + \hat{\tau}_{j_a d})_{a \in A_i^+} \right) \forall i \neq d \right\}.$$

Furthermore, they also indicate that since the functions $\varphi_{id}(\cdot)$ are continuous, concave and component-wise non-decreasing, for each $t \in \mathcal{C}$, we have $t' \in \mathcal{C}$ for any $t' > t$. Therefore, if the free-flow travel time vector $t^0 \in \mathcal{C}$, we have existence of equilibrium.

Since the vector t are commonly defined by road design and existing urban conditions, holding $t \in \mathcal{C}$ actually means defining correctly the functions $\varphi_{id}(\cdot)$, which, in the particular case where these functions are logit-based, specifically means using a bounded set of β_{id} parameters in order to fulfill the condition. These parameters have a key role in the model: they represent the inverse variance value of travel time by choosing a certain arc z_{ad} whose error term has a Gumbel distribution.

3. Integrated stochastic equilibrium model

3.1. Integrated equilibrium model with pure modes choice

In this section we develop a joint equilibrium model for both modes, considering the STE model for transit equilibrium, and the MTE model for traffic equilibrium. Both formulations interact at the demand levels associated with each origin-destination pair and the corresponding travel times, as both modes use the same road infrastructure. Conceptually, the integrated equilibrium model relies on the following premise: some users of the network perform a modal choice at their origin nodes, before starting their trips. These users, who has both modes available, decide whether to use the car or the bus, staying on the traffic network or being transferred to the transit network. In turn, users who have only the bus as the available option, are *captive users* of the bus and move only through the transit network.

The modal choice for users with car and bus availability is conducted by means of a logit function, which takes into account the generalized costs of traveling in both networks, function denoted U_{id}^{m} for origin i and destination d by mode m, with $m \in \{B, C\}$ representing *bus* and *car* respectively. The generalized cost or utility function may contain expected travel times τ_{id}^{m} of the respective network, the monetary costs of using each mode -fares, fuel prices, etc.- and other issues involving modal choice, such as comfort, modal attractiveness or traffic congestion.

Let G_{id}^{B} be the variable denoting the number of users who only have the bus available to travel between the OD pair id, and let G_{id}^{C} be the variable denoting the number of users that can freely choose between the car or the bus to perform the trip on that OD pair. Then, on equilibrium, the input demand on both transit and traffic networks are respectively:

$$g_{id}^{B*} = G_{id}^{B} + G_{id}^{C} \cdot \frac{e^{U_{id}^{B*}}}{e^{U_{id}^{B*}} + e^{U_{id}^{C*}}} \tag{13}$$

$$g_{id}^{C*} = G_{id}^{C} - G_{id}^{C} \cdot \frac{e^{U_{id}^{B*}}}{e^{U_{id}^{B*}} + e^{U_{id}^{C*}}}. \tag{14}$$

Note that expression $G_{id}^{C} \cdot \frac{e^{U_{id}^{B*}}}{e^{U_{id}^{B*}} + e^{U_{id}^{C*}}}$ represents the total number of users that observe travel costs in both modes, but prefer to use the bus, making a transference from the traffic to the transit network. Graphically, the modal split process is shown in Fig. 2.

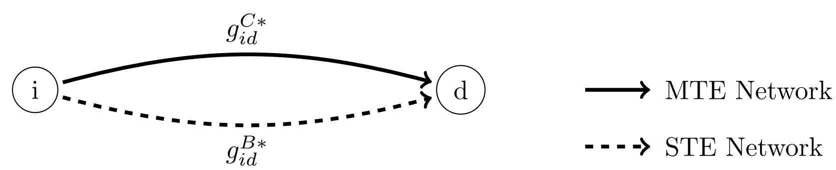


Figure 2: Modal split on traffic and transit networks with pure modes

3.2. Integrated equilibrium model with combined modes choice

The natural extension of the previous model is the addition of the combined option of traveling through both networks during the realization of an OD trip. On our proposal, we consider that a traveller with car and bus availability may choose to drive in the first part of the trip, then to park the car at an intermediate transfer station, and from there, to complete the trip by public transportation towards the final destination. This modality of travel is known as *park and ride*, with the adequate facilities for parking, system that successfully operate in many cities over the world (e.g. Boston, New Jersey, Oxford, Montreal, Norwich, Bristol). Normally, the transfer stations are located on the suburbs of the city, places that should have good connectivity to the public transportation network, thus improving the use of transit systems within the city and consequently reducing the number of private cars within congested urban zones.

In this case, we decide to use a hierarchical logit formulation to represent the modal choice process with combined modes (García and Marín, 2005); the decision levels are two: first, at the upper level the decision is performed by choosing the mode, which can be either car only, bus only, or car and bus as a combined mode. If the traveller chooses the combined option, then there is a second (lower) level decision, for which the user must choose the location of the transfer node between car and bus, among a predefined set of options. The reason for having two decision levels (and therefore, not just analyzing at the same level all possible options between modes and transfer stations), is because there is a strong correlation between the combined mode options, thus the assumption underlying the independent alternatives behind the multinomial logit model does not apply in this case.

Let us denote mode $m \in \{B, C, P\}$ for *bus*, *car* and *combined mode* respectively, and let $K_P \subset D$ be the set of transfer nodes within the transportation network, allowable to combine between car and bus. Then, we denote by $g^P_{id,k}$ the total number of users who travel between the origin i and destination d using the combined mode P, and choosing transfer node $k \in K_P$. Graphically, the demand split for those travelers with both pure modes available is shown in Fig. 3, where each level represents a decision that must be made by the traveller.

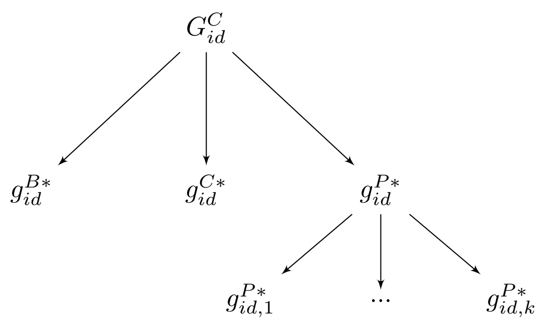


Figure 3: Decision tree of modal and transfer node choice for users with car and bus availability

In the same way we did for the integrated model with pure modes, in this combined formulation there is a primary mode choice, considering the generalized costs of each option. Let γ^m_{id} be the probability of choosing mode m to travel between the OD pair id. Hence,

$$\gamma^m_{id} = \frac{e^{U^{m*}_{id}}}{\sum_{m' \in \{B,C,P\}} e^{U^{m'*}_{id}}}. \tag{15}$$

Generalized costs in the case of pure car or bus modes could become the same formulation used in the integrated equilibrium model just with pure modes. However, for the combined mode, we have to estimate an expected minimum generalized cost function considering all transfer nodes. Let $U^{P}_{id,k}$ be the generalized cost of using the combined mode between the pair id using transfer node k. This function could depend, in addition to the expected travel times in both car and bus segments of the trip, also on other aspects such as parking safety, comfort, accessibility, parking fare, etc. Then, an estimate of the minimum generalized cost among all transfer nodes, which finally is the representative cost of choosing the combined mode at the upper level of the decision tree, can be found using a log-sum formulation:

$$U^{P*}_{id} = \frac{1}{\beta_P} \ln \left(\sum_{k \in K_p} e^{\beta_P \cdot U^{P*}_{id,k}} \right), \tag{16}$$

where parameter β_P indicates the degree of correlation among different parking alternatives to choose from in a hierarchical logit scheme.

Given that a traveler has chosen the combined mode, the conditional probability of choosing k as transfer node can be computed as:

$$\gamma^{P}_{id,k} = \frac{e^{\beta_P \cdot U^{P*}_{id,k}}}{\sum_{k' \in K_P} e^{\beta_P \cdot U^{P*}_{id,k'}}}. \tag{17}$$

Finally, the demand between the pair id on traffic and transit networks are respectively:

$$g^{B*}_{id} = G^{B}_{id} + \gamma^{B}_{id} \cdot G^{C}_{id} \tag{18}$$

$$g^{C*}_{id} = \gamma^{C}_{id} \cdot G^{C}_{id} \tag{19}$$

and the total number of travelers who choose the combined mode between the id pair, transferring at the k transfer node is:

$$g^{P*}_{id,k} = \gamma^{P}_{id,k} \cdot \underbrace{\gamma^{P}_{id} \cdot G^{C}_{id}}_{g^{P*}_{id}}. \tag{20}$$

Graphically, the modal split for traffic and transit networks in the integrated equilibrium model with combined modes is shown in Fig. 4.

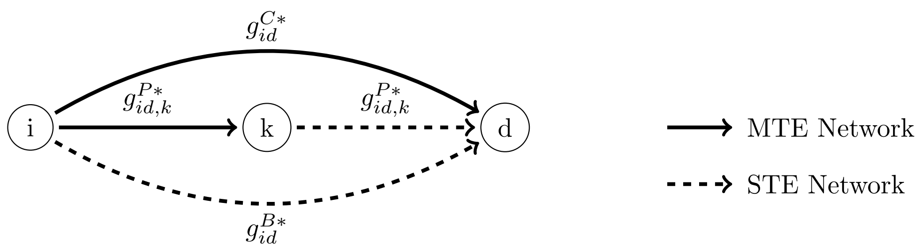


Figure 4: Modal split on traffic and transit networks with combined modes

4. Implementation and results

In this section, an efficient algorithm is proposed for solving the integrated equilibrium model; the procedure is based on the Method of Successive Averages (MSA). The model is iterative. At the beginning of each iteration, apart from computing travel times and effective frequencies in both networks, demand levels are obtained by calculating respective utility functions for each mode,considering the levels of service obtained in the previous iteration as inputs for the next. The algorithm is designed to reach the equilibrium for STE and MTE models separately, which computationally can be solved in parallel and not necessarily through a sequential execution. The algorithm compares the resulting vector of flows with respect to those flows obtained in the previous iteration until reaching a predefined convergence criteria based on similarity of the flows.

Before discussing the integrated equilibrium model algorithm, in the following section we describe a required auxiliary construction procedure for adding walking, boarding and alighting explicitly in the transit network, representation that is necessary for implementing the STE module.

4.1. STE Implementation

4.1.1. Extended transit network

The STE model contains cross dependencies between expected travel time τ_{id}, the conditional probability of boarding a certain bus p_a^d, effective frequencies f_a, travel time t_a and destination flows v_{ad}. Due to such dependences, it turns out that the STE implementation becomes in essence the solution of a fixed point problem among these values, which are interrelated in Definition 1 and Eq. (1) of flow conservation at each node. In order to track resulting destination flows within the transit network as well as to verify flow conservation constraints, we describe with much detail the operation of passengers when boarding or alighting from(to) a bus. In order to describe these processes, we have to add auxiliary nodes and arcs to the original network. This resulting network is called *extended transit network*.

The transit network is represented by three types of nodes: centroids, stop nodes and line nodes. *Centroids* $d \in D$ represent urban areas where demand is generated and attracted; *stops nodes* $s \in N_S$ represents the locations where passengers are allowed to board and alight. Finally, for every line l stopping at node s, we add an additional *line node* $h_s^l \in N_L$ that represent the transit lines available at each stop node. The entire set of extended network nodes is denoted by $N = D \cup N_S \cup N_L$.

For connecting these different types of nodes, we use three different classes of arcs: walking arcs, boarding/alighting arcs and service arcs. Walking arcs (d, s) and (s, d) represent the walking distance between each centroid d and each bus stop s. We include these arcs only for zones and bus stops that are geographically close. The frequencies of such arcs are fixed as $f_a = \infty$ as these arcs are always available once the passenger decided to go to a certain stop, and have no waiting time associated. The travel time on these arcs is equal to the walking time between the zone and the stop.

The second set of arcs are boarding arcs (s, h_s^l) and alighting arcs (h_s^l, s) for each line l stopping at $s \in N_S$. In boarding arcs, the frequency is equal to the effective frequency f_a that the line has on that stop, while travel time t_a in these arcs is assumed negligible and equal to zero, although the model is flexible enough to set these variables in a greater than zero value, including passengers transfer time at stops. Alighting arcs travel time are set as $t_a = 0$ as well, but frequencies are fixed to $f_a = \infty$, as a passenger does not have to wait for alighting the bus, once it stopped. Finally, the third set of arcs represent the on board travel time on each transit line segment. For there, arcs $(h^l_s, h^l_{s'})$ are included for each transit segment of line l connecting nodes s and s', with travel times corresponding to those computed between consecutive stops on the original network, and setting frequency as $f_a = \infty$. Graphically, the representation of a stop in the STE model is shown in Fig. 5.

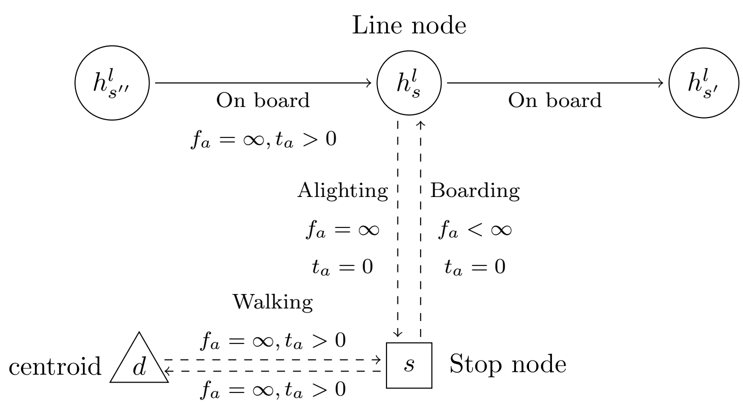


Figure 5: Stop representation on STE extended network

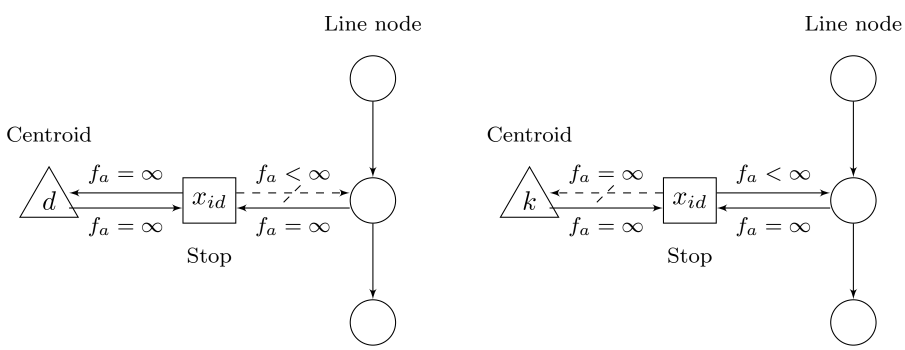
Note that in case of line nodes, all outgoing arcs from those nodes -alighting and on board arcs- have infinite frequency in our representation; the same property applies to walking arcs associated with centroids. In these cases, the effective frequencies values contained in Definition 1 are replaced by the value f_∞, where the limit of those equations as f_∞ approaches infinity, i.e., for all $a \in A_i^+$ and $f_a(v^*) \to \infty$ we compute expressions of Def. 1 as:

$$\tau^*_{id} = \lim_{f_\infty \to \infty} \frac{1 + \sum_{a \in A_i^+} p^d_a f_\infty \left(t_a(v^*) + \tau^*_{j_a d} \right)}{\sum_{a \in A_i^+} f_\infty p^d_a} = \frac{\sum_{a \in A_i^+} p^d_a \left(t_a(v^*) + \tau^*_{j_a d} \right)}{\sum_{a \in A_i^+} p^d_a}, \tag{21}$$

$$v^*_{ad} = \lim_{f_\infty \to \infty} x_{id}(v^*) \frac{f_\infty p^d_a}{\sum_{a \in A_i^+} f_\infty p^d_a} = x_{id}(v^*) \frac{p^d_a}{\sum_{a \in A_i^+} p^d_a}, \tag{22}$$

$$p^d_a = \varphi_a \left(t_a(v^*) + \tau^*_{j_a d} - \tau^*_{id} \right). \tag{23}$$

Moreover, stop nodes combine finite and infinite frequencies on their outgoing arcs. In order to use the equations in the Definition 1, we state the following consideration in the algorithm: when computing the equilibrium for destination d, we remove all boarding arcs from stop nodes adjacent to d, since it is assumed that passengers are rational and they will not ride another bus: these passenger will walk towards the destination instead. Conversely, if none of the centroids adjacent to the stop is the destination d, then walking arcs outgoing from this node are not considered, forcing passengers to boarding another bus for reaching their destinations. Graphically, these modifications are shown in Fig. 6, in which dashed arcs are not considered at each case. Note that after these modification, for each node all incoming/outgoing arcs have either infinite or finite frequency, allowing to compute travel times and induced flows. We denote by $\mathcal{A}$ the resulting set of arcs on the extended network.



Adjacent centroid is destination d Adjacent centroid is not destination d

Figure 6: STE network modifications for Stop nodes when computing the equilibrium for destination d

4.1.2. Computing Expected Travel Time

Expected travel times τ_{id}^{B} are obtained by solving a system of linear equations. At each node $i \in N$ of the extended network, and for each destination $d \in D$, we have the expression for τ_{id} of Definition 1 or Eqs. (21-22), depending on the type of node. In the case where i is a stop node and the destination d is not adjacent to i, by rearranging terms of the Def. 1, we obtain:

$$\sum_{i\in A_i^+} f_a(v)p_a^d(\tau_{id}^B - \tau_{j_ad}^B) = 1 + \sum_{i\in A_i^+} f_a(v)p_a^d t_a(v), \qquad \forall i \in N_S \text{ with } (i,d) \notin \mathcal{A}. \tag{24}$$

If i is a line, centroid or stop node adjacent to destination d, then for all outgoing arcs of i it holds that $f_a(v) \to \infty$, and redefinition for τ_{id} of Eq. (21) applies. Rearranging terms of the latter:

$$\sum_{i\in A_i^+} p_a^d(\tau_{id}^B - \tau_{j_ad}^B) = \sum_{i\in A_i^+} p_a^d t_a(v), \qquad \forall i \in D, N_L \text{ and } \forall i \in N_S \text{ with } (i,d) \in \mathcal{A}. \tag{25}$$

As we can see, Equations (24) and (25) form a sparse system of linear equations of size $|N|\times|N|$, that can be efficiently solved even for large scale networks, obtaining the values τ_{id}^B for a given flow vector v.

4.2. Integrated equilibrium algorithm

Since the integrated equilibrium model with pure modes is a particular instance of the more general model with combined modes, the implementation of an algorithm to solve the equilibrium in a multimodal network will be based on the latter. The structure used to implement the algorithm is the following: we use an iterative MSA-based algorithm, where at first, expected travel time matrices in each network are computed; then, the levels of demand on each network through a modal split model is determined; and finally, with the obtained demands, we solve equilibrium sub-models in each mode. A stop criteria based on similarity of flows vector is proposed.

The general integrated equilibrium algorithm is described in Algorithm 1. STE and MTE modules are described step by step in Algorithms 2 and 3 respectively. Note that for the integrated equilibrium model with pure modes implementation, defining transfer nodes subset as $K_P = \emptyset$ is enough.

Sometimes, the MSA convergence is not monotonic. This occurs because the descent direction may point in a direction such that the norm in some iterations

Algorithm 1 Integrated equilibrium model with combined modes

1: Set initial feasible assignment $v^{C,0}$ y $v^{B,0}$ in both networks
2: Set $n \leftarrow 0$.
3: **repeat**
4: Set $n \leftarrow n+1$.
5: Modal split model: $g_{id}^{B,n}, g_{id}^{C,n}, g_{id,k}^{P,n}$
6: Compute effective frequencies $f_a^n = f_a(v^{B,n-1})$.
7: Compute travel time $t_a^n = t_a(v^{n-1})$.
8: **Solving transit equilibrium by STE** (see Alg. 2)
9: **Solving traffic equilibrium by MTE** (see Alg. 3)
10: **until** $\frac{\sqrt{\sum_a (\bar{v}_a^{B,n+1} - \bar{v}_a^{B,n})^2}}{\sum_a \bar{v}_a^{B,n}} \leq \epsilon$ and $\frac{\sqrt{\sum_a (\bar{v}_a^{C,n+1} - \bar{v}_a^{C,n})^2}}{\sum_a \bar{v}_a^{C,n}} \leq \epsilon$.

increases; it could also happens because the MSA step, given by the value α_n, is fixed a priori, and that may exceed the optimal descent weight (Sheffi, 1985). A convergence criteria which is in general monotonically decreasing can be obtained by averaging the flows over the past q iterations. In this case, if $\bar{v}_a^n$ denotes the average flow in the iteration n:

$$\bar{v}_a^n = \frac{1}{q}\left(v_a^n + v_a^{n-1} + ... + v_a^{n-q+1}\right) \tag{26}$$

then, the convergence criteria may be based on flows similarity for the last q iterations. For example:

$$\frac{\sqrt{\sum_a (\bar{v}_a^{n+1} - \bar{v}_a^n)^2}}{\sum_a \bar{v}_a^n} \leq \epsilon. \tag{27}$$

The latter is the convergence criteria implemented in Algorithm 1.

4.3. Numerical Experiments

In this section, we analyze some numerical experiments by observing the behavior of the integrated equilibrium model. First, the pure modes algorithm is implemented in a simple network, focusing on transit equilibrium and comparing the results among the stochastic model (Cortés et al., 2013) and the

Algorithm 2 STE module for integrated equilibrium model

for all destination $d \in D$ **do**
  Set $l \leftarrow 0$.
  Compute initial conditional probabilities $p_a^{d,0} = \varphi_i^d(t_a^n + \tau_{j_a d}^{B,n} - \tau_{id}^{B,n})$.
  repeat
    Set $l \leftarrow l + 1$.
    Solve system of linear equations for expected travel time

$$\tau_{id}^{B,l} = \frac{1 + \sum_{a \in A_i^+} f_a^n p_a^{d,l-1} \cdot (t_a^n + \tau_{j_a d}^{B,l})}{\sum_{a \in A_i^+} f_a^n p_a^{d,l-1}} \quad \forall i \in N_S \text{ with } (i,d) \notin \mathcal{A}$$

    or,

$$\tau_{id}^{B,l} = \frac{\sum_{a \in A_i^+} p_a^{d,l-1} \cdot (t_a^n + \tau_{j_a d}^{B,l})}{\sum_{a \in A_i^+} p_a^{d,l-1}} \quad \forall i \in D, N_L \text{ and } \forall i \in N_S \text{ with } (i,d) \in \mathcal{A}$$

    Compute conditional probabilities $p_a^{d,l} = \varphi_i^d(t_a^n + \tau_{j_a d}^{B,l} - \tau_{id}^{B,l})$.
  until $\frac{||\tau_{id}^{B,l} - \tau_{id}^{B,l-1}||}{||\tau_{id}^{B,l}||} < \epsilon$.
  Set $p_a^{d,n} = p_a^{d,l}$
  Compute induced flows

$$\hat{v}_{ad} = x_{id}^B \frac{f_a^n p_a^{d,n}}{\sum_{a \in A_i^+} f_a^n p_a^{d,n}} \quad \forall i \in N_S \text{ with } (i,d) \notin \mathcal{A}$$

  or,

$$\hat{v}_{ad} = x_{id}^B \frac{p_a^{d,n}}{\sum_{a \in A_i^+} p_a^{d,n}} \quad \forall i \in D, N_L \text{ and } \forall i \in N_S \text{ with } (i,d) \in \mathcal{A}$$

end for
Update transit flow assignment $v^{B,n} = (1 - \alpha_n) v^{B,n-1} + \alpha_n \hat{v}$.

deterministic version (Cominetti and Correa, 2001; Cepeda et al., 2006). Then, the integrated equilibrium model with the combined modes algorithm is tested on a real transportation network: the city of Iquique, Chile.

Algorithm 3 MTE module for integrated equilibrium model

```
for all destination $d \in D$ do
    Set $l \leftarrow 0$.
    Set $z^0_{ad} = t^n_a$
    Set $\tau^{C,0}_{id} = 0$
    repeat
        Set $l \leftarrow l + 1$.
        Compute expected travel time $\tau^{C,l}_{id} = \varphi_{id}(z^{l-1}_{ad})$
        Compute expected travel time by arcs $z^l_{ad} = t^{C,n}_a + \tau^{C,l}_{j_a d}$.
    until $\frac{||\tau^{C,l}_{id} - \tau^{C,l-1}_{id}||}{||\tau^{C,l}_{id}||} < \epsilon$.
    Compute probabilities
```

$$p^{d,n}_a = \frac{\partial \varphi_{id}(z^l_{ad})}{\partial z^l_{ad}}$$

```
    Compute induced flow
```

$$\hat{v}_{ad} = x^C_{id} \frac{p^{d,n}_a}{\sum_{a \in A^+_i} p^{d,n}_a} \qquad \forall i \in D, N_S$$

```
end for
Update traffic flow assignment $v^{C,n} = (1 - \alpha_n) v^{C,n-1} + \alpha_n \hat{v}$.
```

4.3.1. Stochasticity in STE Model

The Sioux Falls city coding provided by Bar-Gera (2011) was used to analyze the stochasticity effect on the transit network. This original layout corresponds to a traffic network, although we use the same configurations for coding the transit mode, similar to the behavior of an urban subway system.

The Sioux Falls network has 24 nodes, each generating and attracting demand. Therefore, each node is a centroid and a stop node simultaneously. There are 360,000 total trips generated per hour in the system. For testing purposes, we designed 4 subway lines around the city as shown in Fig. 7. Each line l has a nominal frequency of $\mu_l = 30\, trains/hr$ and capacity of $c_l = 1,500\, pax/train$. The conditional probability function used for this experiment is a logistic distribution as follows:

$$p_a^d \equiv \varphi_a^d(\cdot) = \frac{1}{1 + e^{\theta(t_a + \tau_{j_a d} - \tau_{id})}}, \qquad \forall a \in \mathcal{A}; \forall d \in D; \theta \in \mathbb{R}_+. \tag{28}$$

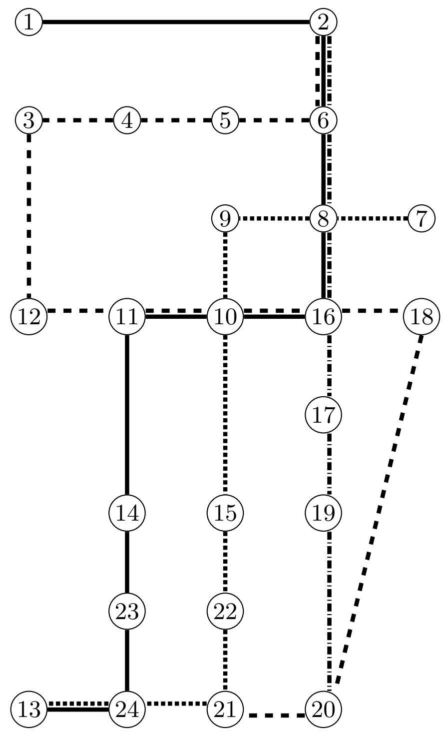


Figure 7: Transit lines design in Sioux Falls

We tested a set of instances for the Sioux Falls network, where the only difference is the parameter θ of the conditional probability p_a^d. Hence, we obtain different stochasticity levels in users decision making process between instances. The formulation of effective frequency is the same as used in Cepeda et al. (2006), with exponent $\beta = 5.0$. The convergence criteria is set at $\epsilon = 10^{-9}$ and the MSA parameter selected for updating the flow is $\alpha_n = \frac{1}{n}$, where n is the number of the current iteration in the general algorithm. The STE equilibrium is reached after around 250 iterations for each instance.

For the analysis of results obtained with the STE model, it is possible to make comparisons between the deterministic and stochastic instances if we define adequate dispersion indicators in equilibrium flows over the network. In principle, in deterministic instances, users choose the minimum hyperpath, causing that the arcs or segments contained in this strategy have flows $v^*_{ad} > 0$, while in those arcs that belong to suboptimal hyperpaths, $v^*_{ad} = 0$. Then, we define the *level of network usage*, denoted ψ^B_d, as the ratio between the number of line segments used in the network to reach the destination d, and the total number of line segments available in the network; while $\bar{\psi}^B$ is the average of level of network usage over all destinations. A *used segment* is such that $v^*_{ad} > 0$. However, as the STE algorithm is a numerical implementation, we consider that a used segment is such that $v^*_{ad} > \xi \approx 0$. In the case of Sioux Falls, $\xi = 0.5[pax/hr]$.

A second interesting indicator in order to analyze flow dispersion in stochastic instances for STE model is the *relative difference* D_d between the flow vector in equilibrium obtained in any stochastic instance with respect to the deterministic case, for destination d. This indicator is calculated as follows:

$$D_d = \frac{\|\vec{v}^{det}_d - \vec{v}^{ins}_d\|}{\|\vec{v}^{det}_d\|}, \tag{29}$$

where $\vec{v}^{det}_d$ is the destination-flow vector resulting in the deterministic case, and $\vec{v}^{ins}_d$ is the resulting equilibrium destination-flow vector for any other instance. Moreover, it is also possible to apply this indicator on the total flow vector, in order to observe general changes in the network:

$$\bar{D} = \frac{\|\vec{v}^{det} - \vec{v}^{ins}\|}{\|\vec{v}^{det}\|}. \tag{30}$$

The deterministic instance in the transit equilibrium for Sioux Falls is obtained by setting $\theta = 30$. The results for different instances, along with the above indicators, are shown in Table 1. In addition, as a measure of the influence of stochasticity on other variables in the model, we include the value of the equilibrium expected time $\tau^{B*}_{1,10}$ from node 1 to node 10, which increases gradually while increasing STE model stochasticity. This result is expected, as stochasticity increases, users integrate other suboptimal strategies for travel between this OD pair and have travel times larger than the minimum obtained for the deterministic case. Therefore, the expected time collects this effect, increasing its value.

We also want to test the null hypothesis that the pairwise difference between the flows in any instance and the deterministic equilibrium has a mean equal to zero (paired-sample t-Test), at the default 5% significance level. Results of applying the test to destination 10 as well as to all destinations are reported in two columns (p-value) of table 1.

Table 1: STE results & indicators for Sioux Falls network

θ	$\bar{\psi}^B$	$\tau^{B*}_{1,10}[min]$	D_{10}	p-value	$\bar{D}$	p-value
30.0	0.44	28.73	0.00	–	0.00	–
15.0	0.44	28.73	0.00	0.709	0.00	0.761
8.0	0.46	28.74	0.01	0.706	0.00	0.758
4.0	0.49	28.82	0.01	0.594	0.01	0.538
2.0	0.65	29.22	0.03	0.621	0.05	0.476
1.0	0.93	30.14	0.06	0.526	0.09	0.221
0.5	0.96	32.20	0.11	0.025	0.14	< 0.001
0.3	0.96	35.67	0.21	< 0.001	0.29	< 0.001
0.2	0.96	40.77	0.32	< 0.001	0.51	< 0.001

Another conclusion from these results is that the reduction of the θ parameter causes the increase in the average level of network usage for all destinations $\bar{\psi}^B$ and a raise of the dispersion of equilibrium flows. This second effect is captured by the increase of the relative difference $\bar{D}$ and can be observed as well at destination-flows level, as shown by the increase of indicator D_{10}. In the last instance, the dispersion flow is so high that, on average, nearly all of transit segments of the network are used by a flow-destination volume greater than ξ. The results of the paired-sample t-test are consistent with the analysis of the other indicators, showing that the flows patterns are statistically different for values of θ below one.

Graphically, the level of network usage for destination $d = 10$ between deterministic and a highly stochastic instances is shown in Figs. 8 and 9.

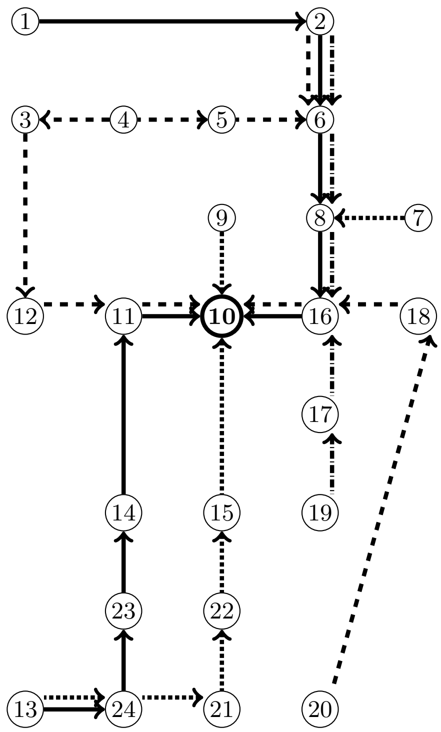


Figure 8: Transit segments used at STE deterministic instance in Sioux Falls. $\theta = 30$ and $\psi_{10}^{B} = 0.46$

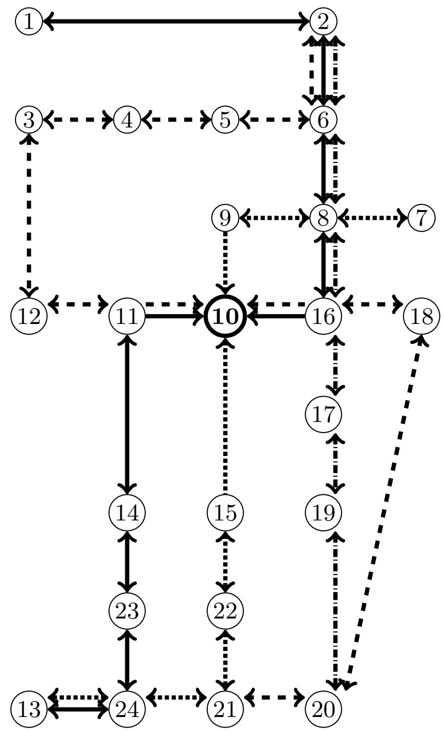


Figure 9: Transit segments used at STE stochastic instance in Sioux Falls. $\theta = 0.5$ and $\psi_{10}^{B} = 0.91$

Another interesting comparison to perform is between the deterministic and stochastic instances associated with the STE model with variation in demand levels. Cortés et al. (2013) show that on a very small 3-nodes network, when the demand generates low flow levels on line segments under equilibrium conditions, the flows difference between the stochastic instance and the deterministic one is highly relevant. However, for high levels of demand, there are no relevant changes in equilibrium flows between both instances. In this section, we perform the same comparison but on the Sioux Falls network, for different levels of demand. The demand variation is obtained by applying a set of amplifying factors for all OD pairs.

The results are shown in Fig. 10, for different levels of transit demand and

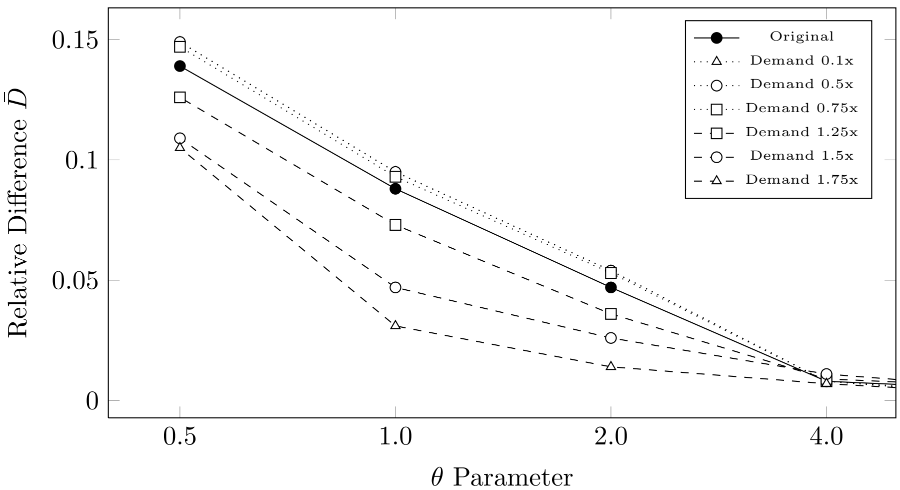


Figure 10: Relative difference $\bar{D}$ with variation of demand and stochasticity in STE instances for Sioux Falls

stochasticity. We can observe the following tendency: when demand increases progressively, the index associated with the relative difference $\bar{D}$ of the equilibrium flow vector between the deterministic and stochastic instances decreases. This concur with the results of Cortés et al. (2013), in a more realistic medium-sized network.

Intuitively, in highly congested networks, users waiting for a bus at a stop see no differences in effective frequencies between lines, because all of them are congested, and all of these values are close to zero. Hence, the flow vector differences between the deterministic and stochastic instances become less relevant, since the operation of the networks are more similar.

4.3.2. Integrated Equilibrium Model with Pure Modes

In the experiment, we analyze the integrated equilibrium model including pure modes only; a private car transport network of Sioux Falls city is used (Bar-Gera, 2011), containing 24 centroids, and 76 unidirectional arcs, and total demand of 360,000 $trips/hr$. Moreover, the public transport network and simulation parameters are identical to those used in testing the STE algorithm, described in section 4.3.1.

The goal of this experiment is to measure the impact of the stochasticity in both the MTE and the STE models. For accomplishing that, a logistic formulation is used in the transit model for the conditional proba bility p_a (Eq. 28), varying the parameter θ. In the case of private cars, a log-sum formulation was chosen for computing the expected minimum travel time τ_{id}^{C}, in which case the parameter β is varied to get different levels of stochasticity. Notice that the modal utility functions are based only on expected travel times for each network, i.e.: $U_{id}^{C} = -\tau_{id}^{C}$ and $U_{id}^{B} = -\tau_{id}^{B}$.

The general results obtained by the integrated equilibrium model algorithm with pure modes for the Sioux Falls network are shown on Tables 2 and 3. The results summarize modal split for users with car and bus availability, and network usage indicators.

Table 2: Integrated equilibrium model with pure modes results for Sioux Falls network, STE stochasticity

θ STE	β MTE	Car Share	Bus Share	$\bar{\psi}^{B}$	$\bar{\psi}^{C}$
30.0	12.0	67.8%	32.2%	0.43	0.48
15.0	12.0	67.8%	32.2%	0.44	0.48
8.0	12.0	67.8%	32.2%	0.44	0.48
4.0	12.0	67.8%	32.2%	0.46	0.48
2.0	12.0	68.2%	31.8%	0.49	0.48
1.0	12.0	69.4%	30.6%	0.65	0.49
0.5	12.0	72.3%	27.7%	0.93	0.49
0.3	12.0	76.9%	23.1%	0.96	0.50
0.2	12.0	81.7%	18.3%	0.96	0.52

As seen in Table 2 increasing stochasticity in the STE model, and keeping a deterministic formulation in the MTE model, causes that users tend to choose the car in higher proportion, because the expected travel time computation in

Table 3: Integrated equilibrium model with pure modes results for Sioux Falls network, MTE stochasticity

θ STE	β MTE	Car Share	Bus Share	$\bar{\psi}^B$	$\bar{\psi}^C$
30.0	12.0	67.8%	32.2%	0.43	0.48
30.0	6.0	67.9%	32.1%	0.43	0.49
30.0	3.0	68.0%	32.0%	0.43	0.49
30.0	1.5	68.0%	32.0%	0.43	0.57
30.0	1.0	67.8%	32.2%	0.43	0.69
30.0	0.5	66.6%	33.4%	0.44	0.98
30.0	0.4	65.8%	34.2%	0.44	1.00
30.0	0.35	64.9%	35.1%	0.44	1.00

STE model integrates suboptimal strategies, increasing these values. Hence, users who have modal choice availability tend to choose the mode that is more predictable in relation to travel times. Furthermore, the same results indicate that the network usage in the STE model, strongly increases as the stochasticity increases, from 0.43 in the deterministic instance to 0.96 in the fully stochastic instance.

A similar analysis can be performed by observing results in Table 3 when increasing stochasticity in the MTE model and a deterministic formulation remaining in STE. In this case, increasing the stochasticity of the MTE causes a higher proportion of users choosing public transport, due to less variability in expected travel times, however changes in modal split are lower compared to the previous case. On the other hand, stochasticity in the MTE model increases the use of arcs in the private transport network so that, on average, the flow for a given destination is dispersed through all available arcs.

4.3.3. Integrated Equilibrium Model in Large Networks

The test scenario to be studied, in the context of our integrated equilibrium model including the option of combining modes, is a real network representation of the city of Iquique in the north of Chile. The real transit network contains 72 centroids, 485 bus stops and 2,118 unidirectional transit line segments. The network belongs to a weekday morning peak period, calibrated for the year 1998, with a total of 5,449 $trips/hr$ during such a period. The size of the STE extended network for this city is 2,711 nodes -including centroids, stops and line nodes-, and 7,144 arcs -including walking, boarding, alighting and on board arcs-.

On the other hand, the traffic network includes 72 centroids, 485 intersection nodes and 2,180 unidirectional road arcs, which are identical to those contained in the transit network, using the same road infrastructure, for a total of 10,646 $trips/hr$.

The integrated equilibrium model requires to define modal utility functions and parameters, in order to obtain the modal split of those users that have car, bus and combined mode availability. The network coding, modal utility functions and some calibrated parameters for the modal split model in Iquique were provided by the Ministry of Transport and Telecommunications of the Chilean Government. We will use the following utility function U_{id}^{C}, representing the disutility of passengers for a user traveling from i to d using car:

$$U_{id}^{C} = \theta_C + \theta_{tgen} \cdot \tau_{id}^{C} + \theta_{cost} \cdot \frac{\tau_{id}^{C} c_{unit}}{I} \tag{31}$$

where:

θ_C	car modal constant [utility]
θ_{tgen}	generalized time parameter [utility/min]
θ_{cost}	monetary cost parameter [utility/$]
c_{unit}	car use monetary cost per unit time [$/min]
I	income level [$]

In addition, the utility function for passengers using bus is:

$$U_{id}^{B} = \theta_B + \theta_{tgen} \cdot \tau_{id}^{B} + \theta_{cost} \cdot \frac{c_{bus}}{I} \tag{32}$$

where:

θ_B	bus modal constant [utility]
c_{bus}	bus fare [$]

The remaining parameters for the bus utility function are the same as defined in the case of the car on Eq. (31), and for simulation scenario performance, most of all parameters for Iquique, showed in Table 4 were previously calibrated.

Table 4: Modal utility parameters for Iquique, year 1998

Parameter	Mode		
	Car	Bus	Combined
θ_m	0.502	0.278	-1.500 [1]
θ_{tgen}	-0.023	-0.023	-0.023
θ_{cost}	-43.800	-43.800	-43.800

The combined mode representative utility U_{id}^{P} using a log-sum formulation of Eq. (16) is required. First, we need to compute the value of the utility corresponding to the combined mode, where the interchange between car and bus occurs at parking facility $k \in K_P$, $U_{id,k}^{P}$. For this, we assume that this utility has a functional cost and time structure similar to the sum of utilities of choosing each mode in their respective section of the trip, adding a constant θ_P that represents the modal interchange disutility at the parking lot, which reflects other conditions not included as variables in our model, such as comfort, security, accessibility, infrastructure, etc. In the case of Iquique, this constant was not previously calibrated, so that a consistent value is assumed to perform

[1] Not a calibrated parameter. Defined for simulation purposes

the simulation scenario. Moreover, we will add to the combined mode utility the fare of parking a car c_{park}, as well as the queuing time caused by the other vehicles, when entering the parking lot. The latter term, denoted by T^k_{wp}, is computed as follows: consider that the parking entrance is a queue of type $M/M/r$, where arrivals and service in the parking are assumed to be Markovian, with arrival rates λ_k, together with r available servers, each of them with a service rate μ_k. Then, the average waiting time in a queue with these characteristics can be approximated by (Larson and Odoni, 1981):

$$T^k_{wp} \approx \frac{\frac{\lambda_k}{(r_k\mu_k)^2}}{1 - \frac{\lambda_k}{r_k\mu_k}} + \frac{1}{\mu_k}. \tag{33}$$

Equation (33) has an implicit capacity constraint, since it is only valid when $\frac{\lambda_k}{r_k\mu_k} < 1$. In the algorithm implementation, the arrival rate to the parking lot is equal to the sum of all incoming flows to the centroid that represents the parking, i.e $\lambda_k = \sum_{a \in A^-_k} v_a$. Thus, the utility of choosing the combined mode using the parking lot k as modal interchange location is as follows:

$$U^P_{id,k} = \theta_P + \theta_{tgen} \cdot (\tau^C_{ik} + T^k_{wp} + \tau^B_{kd}) + \theta_{cost} \cdot \frac{(\tau^C_{ik} c_{unit} + c^k_{park} + c_{bus})}{I}. \tag{34}$$

The equilibria achieved by the integrated model with combined modes for Iquique, both on the traffic and transit networks, are graphically shown in Figs. 11 and 12 respectively, with the two original parking lot locations from year 1998. Modal split in equilibrium obtained for users with car available, shows that 67% of the trips are by car, 28% by bus and 5% choose the combined mode. By observing the transit network, the STE model shows high levels of passenger congestion in the south of the city (lower part of Fig. 11), in the arcs close to the parking lots. In those places, users who choose the combined mode make modal interchange from car to buses, which in their majority are going to downtown Iquique, located at the upper left area of Fig. 11. MTE model shows low levels of vehicle congestion in the road network around the city.

The integrated equilibrium model algorithm tested on the described Iquique instance reached a relative similarity of flows for both networks lower than 10^{-5}

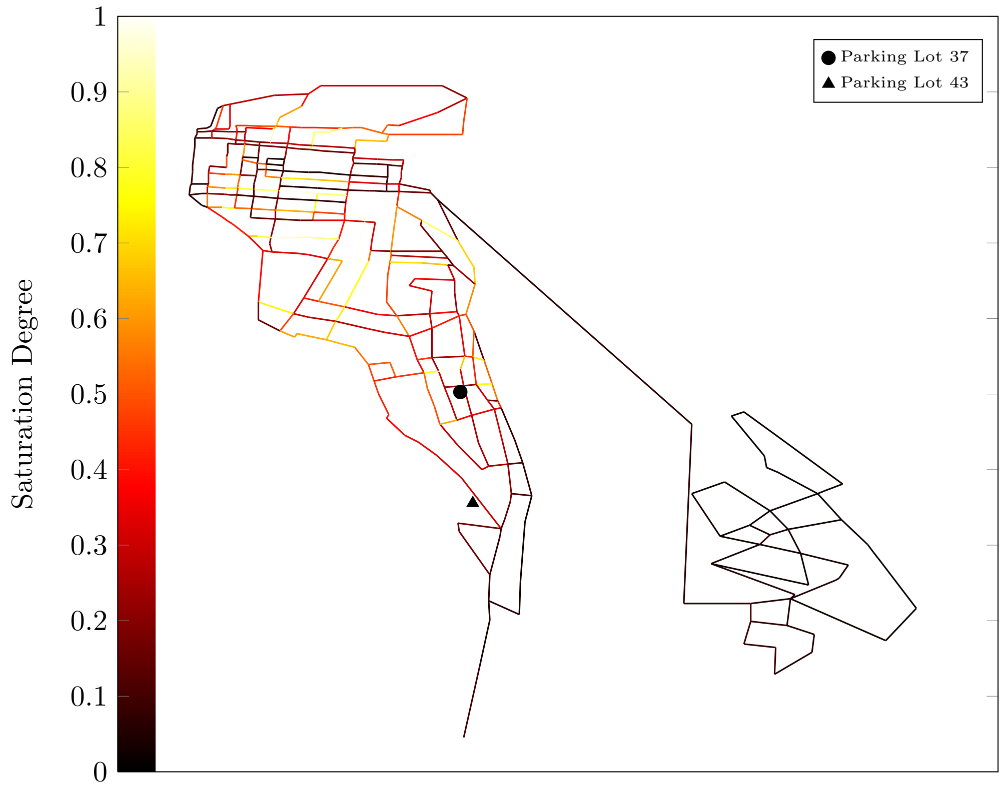


Figure 11: STE equilibrium, Iquique network

after 50 iterations, as shown in Fig. 13, using the flows average from the last 3 iterations as MSA convergence measure. Overall CPU time is 40 min on a 2.4 GHz processor for this instance.

5. Conclusions

In this paper we present an integrated traffic-transit stochastic equilibrium model, based on state of the art equilibrium models for transit and traffic, such as STE (Cortés et al., 2013) and MTE (Baillon and Cominetti, 2008). The model explicitly includes the uncertainty that drivers as well as passengers experiment while they are choosing routes (strategies) until completing their trip, recognizing that within the population the lack of knowledge about conditions and

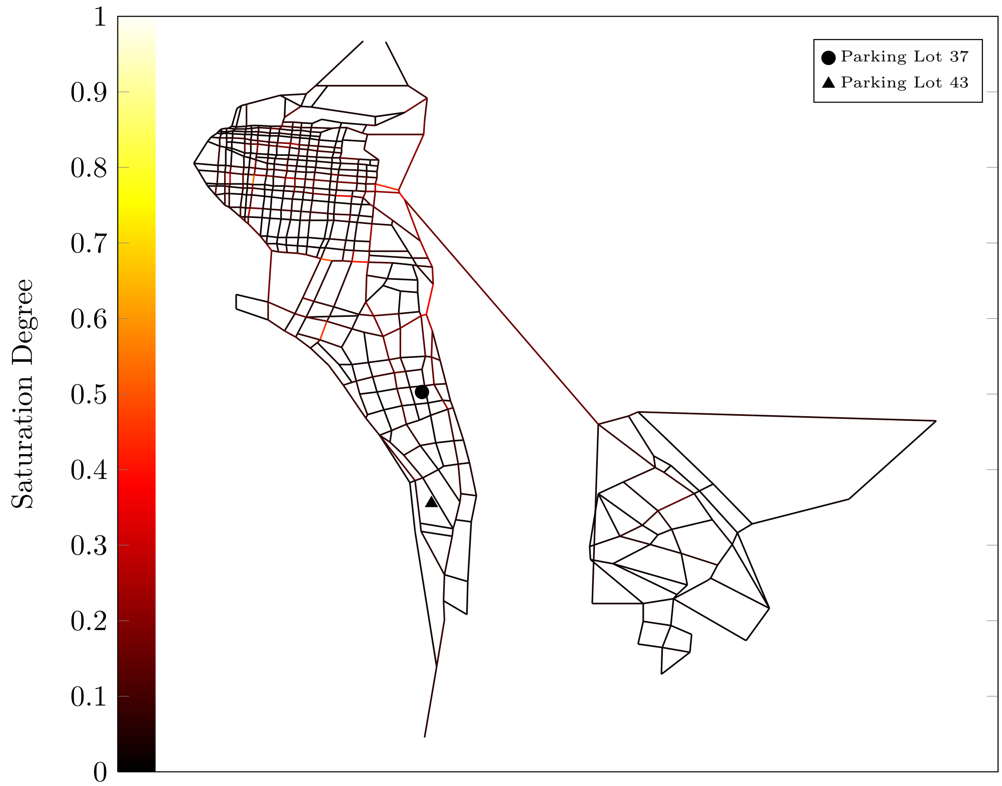


Figure 12: MTE equilibrium, Iquique network

physical characteristics of the networks cause different perceptions among people. It allows us as well to propose an integrated stochastic equilibrium scheme, which captures the interaction between the two modes, recognizing that both cars and buses share the same road infrastructure in many urban areas over the world. Moreover, the integrated equilibrium model reflects the interaction at demand and modal split level, as part of the population has the option to choose between the two modes or even combine them in a park and ride scheme.

We have shown how to apply the STE model to real networks considering their specific characteristics. The major issue is to study the equilibrium conditions of Definition 1 for the cases of nodes that are adjacent to arcs that have infinite frequency. A second issue is how to construct an extended network that

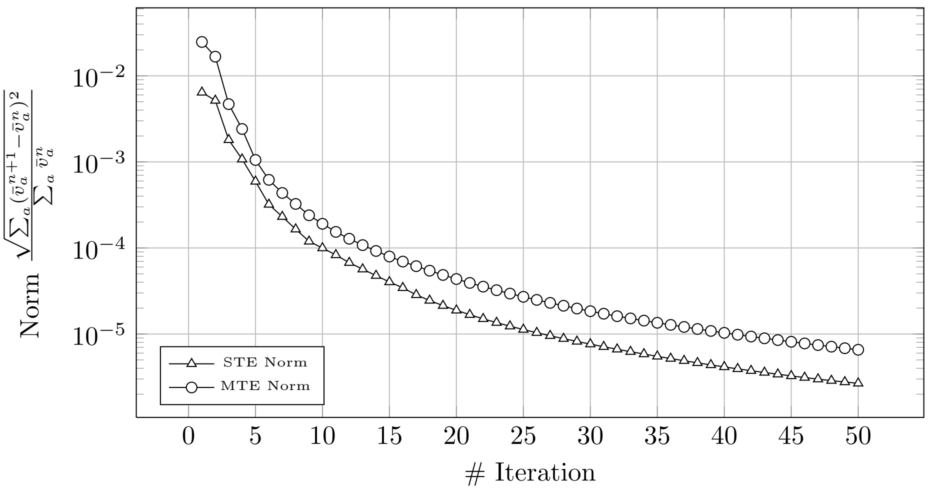


Figure 13: MSA iterations, Iquique network

results from the provided operation pattern of the transit system, incorporating stop nodes, alighting and boarding arcs as well as access arcs that connect centroids to bus stops nodes.

We provide an algorithm to obtain the integrated stochastic equilibrium (for both pure and combined modes cases) and we present numerical experiments on real networks. In these experiments we have observed, as expected, that an increase in stochasticity causes more dispersion on equilibrium flows and an increase in expected travel time. The implementation on a large network, including combined modes, reaches equilibrium flows in a fairly reasonable execution time. In terms of policy issues and implications, we strongly believe that even at a strategic level modeling, the stochastic aspect of human behavior when making daily travel decisions in real urbanizations play a fundamental role, and can make a difference when policy makers decide the investment in different transport projects and plans; then, the development of an efficient tool for modeling all these aspects under an equilibrium scheme considering public and private transport in an integrated scheme can make a major difference in such important decisions for people.

The general algorithm proposed to solve the integrated equilibrium consolidates the formulations and methods of each model individually. One of the major advantages in this sense is that the resolution of partial equilibria is possible to be conducted simultaneously, rather than sequentially, which implies an important computational time reduction and consistency of the final results.

It should be noted that even tough in this article we propose a hierarchical logit modal split formulation between the two systems, the integrated equilibrium model is flexible enough for integrating other modal split proposals, without losing the structural elements of the final joint stochastic equilibria. This issue could be explored as a next step of this research, together with other research questions such as the appropriate formulation of the conditional probability of boarding. Other issues to analyze in the future are related to convergence and uniqueness of solutions for the integrated approach built from results already proved for the separate cases.

In algorithmic terms, the increase in convergence speed to an equilibrium using other methods different from MSA is another topic that requires further study. For example, optimizing the value of the step value between the induced flow and the resulting flow vectors by other numerical methods based on linear search or Newton's method may result in a faster execution of the algorithm.

References

Baillon, J., Cominetti, R., 2008. Markovian traffic equilibrium. Mathematical Programming Series B 111, 33–56.

Bar-Gera, H., 2011. Transportation test problems. URL: http://www.bgu.ac.il/~bargera/tntp/.

Bouzaïene-Ayari, B., Gendrau, M., Nguyen, S., 2001. Modeling bus stops in transit network: A survey and new formulations. Transportation Science 35, 304–321.

Cepeda, M., Cominetti, R., Florian, M., 2006. A frequency-based assignment model for congested transit networks with stric capacity constrains: characterization and computation of equilibria. Transportation Research Part B 40, 437–459.

Chriqui, C., Robillard, P., 1975. Common bus lines. Transportation Science 9, 115–121.

Cominetti, R., Correa, J., 2001. Common-lines and passenger assignment in congested networks. Transportation Science 35, 250–267.

Cominetti, R., Facchinei, F., Lasserre, J., 2012. Modern Optimization Techniques. Birkhäuser.

Cortés, C., Jara-Moroni, P., Moreno, E., Pineda, C., 2013. Stochastic transit equilibrium. Transportation Research Part B 51, 29–44.

Daganzo, C., Sheffi, Y., 1977. On stochastic models of traffic assignment. Transportation Science 11, 253–274.

De Cea, J., Fernández, F., 1993. Transit assignment for congested public transport systems: An equilibrium model. Transportation Science 27, 133–147.

Dial, R., 1971. A probabilistic multipath traffic assignment model which obviates path enumeration. Transportation Research 5, 83–111.

Fernández, E., De Cea, J., Florian, M., Cabrera, E., 1994. Network equilibrium models with combined modes. Transportation Science 28, 182–192.

Fisk, C., 1980. Some developments in equilibrium traffic assignment. Transportation Research Part B 14, 243–255.

Florian, M., 1977. A traffic equilibrium model of travel by car and public transit modes. Transportation Science 11, 166–179.

Florian, M., Los, M., 1979. Determining intermediate origin-destination matrices for the analysis of composite mode trips. Transportation Research Part B 13, 91–103.

Florian, M., Spiess, H., 1983. On binary mode choice/assignment models. Transportation Science 17, 32–47.

García, R., Marín, A., 2005. Network equilibrium with combined modes: models and solution algorithms. Transportation Research Part B 39, 223–254.

Larson, R., Odoni, A., 1981. Urban Operation Research. Prentice-Hall, NJ.

Nguyen, S., Pallotino, S., 1988. Equilibrium traffic assignment for large scale transit networks. European Journal of Operational Research 37, 176–186.

Sheffi, Y., 1985. Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods. Prentice-Hall, NJ.

Spiess, H., Florian, M., 1989. Optimal strategies: a new assignment model for transit networks. Transportation Research Part B 23, 83–102.

Wong, S., 1998. Multi-commodity traffic assignment by continuum approximation of network flow with variable demand. Transportation Research Part B 32, 567–581.